Optimal Land Development Decisions

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This research models the development decision when net rents are growing geometrically and uncertainly, and capital intensity is variable. We derive simple rules for the optimal timing of land development projects based on the commonly used internal rate of return and net present value criteria. We show that, even under certainty, projects are optimally delayed beyond the point where net present value becomes nonnegative, if expected cash flows are growing. The ability to vary capital intensity also raises the specter of perverse responses where increases in interest rates accelerate investment decisions. The positive responses occur when growth rates are high or uncertainty is high.

1. INTRODUCTION

It is now widely recognized that the decision to invest is a decision to exercise a real option and that many insights from the theory of financial options apply to real investment decisions [10]. One of the important differences between real and financial options is the ability to vary the capital intensity of the investment, i.e., the capacity or output level. The capital intensity of a project (as opposed to the scale of a project) is important when there is a fixed factor like land or labor. Analytically, the ability to vary capital intensity means that the exercise price of the option is endogenous rather than fixed. The most common

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application of options with an endogenous exercise price is to real estate development decisions.

In this paper we present a class of land development models where rents grow geometrically and production technology is constant elasticity of substitution (CES). We proceed in a deliberate and didactic manner with a minimum of the stochastic calculus that underpins option theory. In addition, we provide links to traditional urban economic theory (e.g., [2]) and to traditional investment theory (e.g., [13]). By doing so we are able to illustrate three results in an accessible way.

First, what is arguably the most important result of real investment theory, that projects are optimally delayed beyond the naive hurdle (\(\text{IRR} = \text{cost of capital}\)), is also obtained in certainty models when cash flows are growing. The implication is that uncertainty is a sufficient but not a necessary condition for optimal delay. We show that projects are delayed further if capital intensity is variable.

For example, consider the decision to construct a building on a plot of land. If the land is in a growing urban area, rents will be rising. If the owner commits today to the currently optimal intensity, he sacrifices some revenue in the future compared with what he would obtain if he waited and constructed the larger building that becomes optimal next period. This occurs because the optimal capital intensity is positively related to the level of rents and because the investment is irreversible. Thus, the optimal timing of the development balances the revenue lost today from delaying the project against the increase in revenue obtained by building to a higher capital intensity and higher capacity later.

Second, we generalize the optimal decision rules arising from the real options approach (the optimal hurdle rent)\(^2\) and the more familiar rules arising from neoclassical investment theory such as current yield, internal rate of return (IRR), net present value (NPV), and Tobin’s \(q\) ratios for the case where intensity is variable. We contrast the critical values under both certainty and uncertainty. The modified rules indicate how irreversibility and variable intensity alter optimal development decisions under a variety of investment conditions. We show that even though the decision rules in terms of NPV and IRR are the same with variable intensity as with fixed, variable intensity affects the timing of the decision.

The easiest of the traditional rules to apply to development decisions is current yield, which holds that a project is undertaken when the current yield equals the cost of capital. This rule is modified only in the uncertainty case when it is augmented by a simple additive term related to the levels of volatility (uncertainty) and growth.

\(^1\)The comparable tradeoff in real options models is between lost revenue from delay and a reduced probability of losses when a project is undertaken under more favorable conditions (e.g., higher prices).

\(^2\)See Dixit and Pindyck [10, Chap. 6].
Third, we derive the circumstances under which a rise in interest rates leads to a rise in investment. This seemingly perverse result arises because irreversibility uncouples optimal decision rules from a monotone relationship with interest rates alone. When projects are optimally delayed, the NPV is positive and the IRR exceeds the cost of capital at optimal exercise. An increase in interest rates and the cost of capital can be offset by a decrease in the option value that accelerates investment projects. In a growing economy the NPV and IRR on delayed investments are rising over time. An increase in the interest rate may discourage investment by increasing the cost of capital, but may speed optimally delayed investment because the option value of waiting has fallen. On balance the acceleration of projects causes increased investment.

This does not mean that negative NPV projects are being undertaken. It does imply that positive NPV projects are undertaken earlier. Eventually with a growing economy the same projects are commenced. However, during a given period, if interest rates rise investment may increase over what it would have been with steady interest rates.

When interest rates rise, there are two offsetting effects. The first is the increase in the hurdle IRR because the cost of capital is higher. This first effect always occurs. The second is the increase in the project IRR because investment is made with less capital. The second occurs only when capital intensity is variable. When the interest rate rises, the optimal intensity falls since investors will substitute other factors for capital. With a production function exhibiting diminishing returns, the decline in capital intensity will result in an increase in the output/capital ratio, a decrease in the cost of investment per unit of output, and a corresponding increase in the IRR from the project. If the project IRR increases more than the cost of capital, the net effect accelerates some projects.

We show that the critical parameters are the elasticity of substitution in production and the ratio of the growth rate to the interest rate. The positive interest rate response occurs when the ratio of the growth rate to the interest rate is large relative to one minus the elasticity of substitution. High levels of uncertainty about future cash flows are a sufficient but not a necessary condition for this counterintuitive behavior. Similar positive reactions of development to interest rate increases occur under certainty when capital intensity is variable.

These interest rate predictions provide readily testable implications for empirical work. In a companion piece [7] we have tested these hypotheses on a sample of annual data for residential construction in the 1980s. The results are consistent with the theory. Indeed, during the sample period, it is found that 25–50% of the observations lie in the positive response range.

In the next section we outline the general structure of the model of real investment with exponential growth and variable intensity underlying our results. The following two sections discuss the deterministic or perfect foresight case and the stochastic or rational expectations case. Within each of these sections the optimal decision rules for investment are summarized in these sections and
the interest rate sensitivity of investment is derived. The final section concludes and discusses policy implications.

2. THE DEVELOPMENT DECISION

Our point of departure is the general model of the option to replace capital given in Capozza and Li [6]. Here we simplify by allowing a single state variable, $X(t)$, the net rent or cash flow per unit of output from the project. Development projects are new investments; i.e., no durable capital committed in the past will be lost if the project is undertaken. There is one fixed factor, land, and at the time of investment the developer chooses the capital intensity, $K$, and output or capacity level, $Q(K)$.³

The production function, $Q(K)$, is assumed to be increasing and concave. Without loss of generality, a unit of capital is assumed to cost one dollar so that $K$ is also the cost of the investment in the project.⁴ Once the capital cost of the investment is committed to the project, it is assumed to be irreversible.

The net rent or cash flow, $X(t)$, at time, $t$, may evolve stochastically over time following geometric Brownian motion of the form

$$dX/X = gdt + \sigma dz,$$

(1)

where $g$ is the mean growth rate, $\sigma$ is the volatility or standard deviation of the growth rate, and $dz$ is the increment of a standard Wiener process. Decision makers know the parameters of the process so that there is either perfect foresight (when $\sigma = 0$) or rational expectations (when $\sigma > 0$).

At any time $t$, the price of a unit of space is the present value of expected future cash flows,⁵

$$P(X(t)) = E_t \left[ \int_t^\infty X(s)e^{-r(t-s)}ds \right],$$

(2)

where $E_t$ is the expectation conditional on the information about the risk-adjusted cash flow, $X(t)$, and $r$ is the discount rate.⁶ Given the cash flow

³Capital is assumed to be infinitely durable and does not depreciate.
⁴While this assumption simplifies the notation, there is no loss of generality. The results also apply to the more general case in which the unit cost of capital is stochastic as long as the state variable $X(t)$ is viewed as the ratio of unit cash flow to the cost of a unit of capital [14].
⁵The market for space is assumed to be competitive so that the demand for space, $Q(K)$, is perfectly elastic.
⁶We do not model the interest rate process so that our results should be viewed as comparative statics with respect to the interest rate. In effect, we treat interest changes as permanent. This is similar to assuming that interest rates follow a random walk in a stochastic setting. If interest rates mean revert, i.e., changes have a temporary and a permanent component, the rational reactions to the rate changes will be smaller and will depend on the rate of reversion. Since real estate projects are long-lived, the interest rate should be a long term one. Evidence of mean reversion is much weaker in the long rate than in the short rate.
process in (1), the price per unit of space, \( P \), is a function of the cash flow at time \( t \), \( X(t) \). The value of the annual output level \( Q(K) \) is \( Q(K)P(X) \).

If the project is undertaken at time \( t \), the net present value (NPV) of the project is
\[
V(X) = Q(K)P(X) - K
\]
which is the present value of future cash flows minus the cost of the investment.\(^7\)

The investment problem is to choose the number of units of capital, \( K \), and the time of development, \( T \geq t \), to maximize the value of the investment opportunity,
\[
W(X) = \max_{t,K} E_t[X(T)e^{-r(T-t)}].
\]
where \( E_t \) is the conditional expectation defined above, \( T \) is a random first stopping time adapted to the cash flow process, and \( K \) is chosen to maximize the NPV at time \( T \). \( W(X) \) represents the value of a perpetual warrant or option to invest in the project at any future date. It is also the present value at time \( t \) of the NPV at the optimally chosen time of investment.

3. OPTIMAL DECISION RULES WITH PERFECT FORESIGHT

In this section we derive the optimal investment rules under certainty (i.e., the deterministic or perfect foresight case) and show that delaying investment relative to the traditional IRR = cost of capital or \( NPV \geq 0 \) rules is optimal even under certainty.\(^8\) Let the state variable in (1) increase at a constant rate \( g(0 \leq g < r) \) with a zero standard deviation (\( \sigma = 0 \)). Then from (2), the present value of the net cash flow is
\[
P(X) = \frac{X}{r - g}.
\]
Equation (5) is the familiar income capitalization formula used to value income property. Since the cash flow at future time \( T \) is \( X(T) = X(t)e^{g(T-t)} \), the present value factor can be expressed in terms of cash flows,
\[
e^{-r(T-t)} = \left[ e^{-g(T-t)} \right]^{r/g} = \left[ \frac{X(t)}{X(T)} \right]^{r/g}.
\]
From this expression and Eq. (3), the value of the investment opportunity in (4) can be rewritten as
\[
W(X) = V(X^*) \left[ \frac{X}{X^*} \right]^{r/g} = \left[ \frac{Q(K^*)X^*}{r - g} - K^* \right] \left[ \frac{X}{X^*} \right]^{r/g}
\]
for \( X \leq X^* \), where \( X^* = X(T) \) is the threshold or hurdle cash flow for an optimal investment to take place.

\(^7\)When the meaning is clear, we drop the time index for \( X \) so that \( X \) should be understood to be \( X(t) \) if not otherwise specified.

\(^8\)The model in Arnott and Lewis [2] is similar to this deterministic case.
3.1. Optimal Decision Rules

From (3), the first-order condition with respect to capital, \( K \), implies that when the optimal level of capital \( K^* \) is chosen, the marginal benefit of capital equals the marginal cost of capital,

\[
\frac{Q'(K^*)X^*}{r - g} = 1,
\]

(7)

where \( Q' = dQ(K)/dK \). Equation (7) implies that an increase in \( X^* \) must be associated with an increase in \( K^* \) as the production function exhibits diminishing marginal returns \( Q''(K) < 0 \).

Similarly, from (6), the first-order condition with respect to \( X^* \) implies that at time \( T \) the current cash flow yield \( Q(K)X/K \) satisfies

\[
\frac{Q^*X^*}{K^*} = r.
\]

(8)

This equation is similar to the Jorgensonian [13] rule to invest when the cash flow, \( Q^*X^* \), equals the user cost of capital, \( rK^* \). Equations (7) and (8) determine the optimal threshold level, \( X^* \), and the optimal level of capital, \( K^* \).

Note that from (3) and (5), for an optimally chosen level of capital, \( K^* \), the internal rate of return (IRR) for the project is

\[
IRR = \frac{QX}{K} + g.
\]

(9)

From (5), \( g \) is also the rate of appreciation of the price of output. Therefore, the two terms in (9) are the current yield and the capital gains yield. These components are analogous to the dividend yield and capital gains from investing in an asset that pays steadily increasing dividends.

From (8) and (9), investment takes place when the IRR reaches the hurdle rate given by

\[
IRR^* = r + g.
\]

(10)

In capital budgeting theory in finance and neoclassical investment theory in economics, the traditional rule is to invest if the IRR of the project exceeds the cost of capital, \( r \). For irreversible projects with the possibility to delay, if cash flows are rising over time the optimal rule is to delay the project until the current yield, rather than the IRR, equals the cost of capital (Eq. (8)). Thus, the hurdle IRR must include the expected growth rate of cash flows. That is, even under certainty the appropriate hurdle IRR is not equal to the cost of capital.

\footnote{Equation (6) can be rewritten as \( W(X) = X^{\alpha}(Q(K^*)/(r - g))X^{\alpha - 1} - K^*X^{\alpha - 1} \). The first-order condition w.r.t. \( X^* \) is \( (Q(K^*)/(r - g))(1 - r/g)X^{\alpha - 2} - K^*(r/g)X^{\alpha - 2} = 0 \). Simplifying and rearranging, one obtains (8).}
Substituting (5) into (3) and using (8) yields the critical net present value, $V^*$,

$$V^* = \frac{g}{r - g} K^*, \quad (11)$$

which is proportional to the optimal level of capital, $K^*$. Equation (11) indicates that the critical value must reflect the present value of growth in cash flows.$^{10}$

The critical value of Tobin’s $q$, defined as the present value of cash flows per unit of the cost of investment, $q = \frac{Q(X)}{K}$, is

$$q^* = \frac{r}{r - g}, \quad (12)$$

which is always greater than one since $r$ is greater than $g$. The decision rules are summarized in Table 1. Note that these decision rules are independent of the specification of the production function.

3.2. Discussion

The decision rules based on $IRR$ and $NPV$ are illustrated in Fig. 1.$^{11}$ With an option to delay the project, the $IRR$ and the $NPV$ of the project rise over time as the cash flow, $X$, rises. When $X < X^*$, $IRR$ is less than $IRR^*$ and the $NPV$ from investing today is less than the value of the investment opportunity, $W$. In this situation, the value of the investment opportunity includes the NPV from investing today and a positive value of waiting. Optimal investment takes place when $X = X^*$; at that time the $IRR = IRR^*$ and the $NPV = V^*$.

Notice that since $Q = Q(X)$ and $K = K(X)$, $Q/K$ is decreasing in $X$ rather than fixed when intensity is variable. As a result, the optimality condition in Eq. (9) is met at a higher rent level, $X$. Therefore, variable intensity further delays development relative to the exogenous intensity case.

3.3. An Explicit Solution

Dividing (7) by (8) and using (5) implies that the elasticity of output, given by $\gamma(K) = K Q'(K)/Q(K)$, at the optimal level of capital is

$$\gamma(K^*) = 1 - \frac{g}{r}, \quad (13)$$

which satisfies $0 < \gamma(K) < 1$ for $0 < g < r$. If the elasticity of output is decreasing, (13) determines a unique optimal level of capital, $K^*$.

$^{10}$Note that, from (13)–(15), when the production function is Cobb–Douglas ($\rho \to 0$), investment takes place immediately or is delayed forever. The optimal level of capital is either zero or infinity. Thus, reasonable models of real estate development cannot have both geometric growth and Cobb–Douglas production technology.

$^{11}$Since the current yield and Tobin’s $q$ are linear transformations of $IRR$, their graphical illustrations are similar to that of $IRR$. 
TABLE 1
Critical Values for Current Yield, IRR, NPV, and Tobin’s q

<table>
<thead>
<tr>
<th>Variable</th>
<th>Certainty, ( \sigma = 0 )</th>
<th>Uncertainty, ( \sigma &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Benchmark case, ( g = 0 )</td>
<td>( g &gt; 0 )</td>
</tr>
<tr>
<td>Current yield, ( Q'X'/K' )</td>
<td>( r )</td>
<td>( r )</td>
</tr>
<tr>
<td>IRR</td>
<td>( r )</td>
<td>( r + g )</td>
</tr>
<tr>
<td>NPV yield, ( NPV'K' )</td>
<td>( 0 )</td>
<td>( g/(r - g) )</td>
</tr>
<tr>
<td>Tobin’s q, ( Q'P'/K' )</td>
<td>( 1 )</td>
<td>( r/(r - g) )</td>
</tr>
</tbody>
</table>

Note: \( g \) and \( \sigma \) are the mean and the standard deviation of the growth rate. \( r \) is the interest rate. \( \alpha \) is a constant, \( 1 < \alpha < r/g \).

To obtain an explicit solution, we specify the production function to be constant elasticity of substitutes (CES),

\[
Q(K) = [a + (1 - a)K^{-\rho}]^{-1/\rho},
\]

where \( a \) is the asset’s distribution coefficient with \( 0 < a < 1 \) and \( \rho = (1 - \pi)/\pi > 0 \) with \( 0 < \pi < 1 \) being the coefficient of the elasticity of substitution between capital and land. The CES production function exhibits decreasing returns to scale because the elasticity of output for this production function is

\[
\gamma(K) = \frac{1}{1 + (a/(1 - a))K^\rho} < 1.
\]

Since \( \gamma(K) \) is decreasing in \( K \), the optimal level of capital can be determined endogenously. As \( \pi \to 1, \rho \to 0 \), and the production function reduces to the Cobb–Douglas function, \( Q(K) = K^{1-a} \). If \( \pi > 0 \), then this function also exhibits decreasing elasticity of output. This property is necessary to obtain an interior solution when the growth rate of rents is geometric.

From (13) and using the production function, the optimal level of capital \( K^* \) then is

\[
K^* = \left(\frac{1/a - 1}{r/g - 1}\right)^{1/\rho}. \tag{14}
\]

From (8), the threshold level that triggers investment is given by

\[
X^* = r \left[\frac{1 - a}{1 - g/r}\right]^{1/\rho}. \tag{15}
\]

\(^{12}\)This special case has been studied by Williams [16] who assumes that output, \( Q(K) \), is given exogenously. Because the Cobb–Douglas production function exhibits constant elasticity of output as well as constant elasticity of substitution, the optimal intensity is infinite.
3.4. **Discussion**

The optimal level of capital, $K^*$, the critical net present value, $V^*$, and the value of the investment opportunity, $W$, are increasing functions of the growth rate, $g$, and decreasing functions of the level of the interest rate, $r$. These comparative statics results are consistent with those in Capozza and Li [6] where cash flows are assumed to increase linearly. While the effect of the growth rate on the threshold rent level, $X^*$, is ambiguous in Capozza and Li [6], here a higher growth rate always implies a higher threshold level and more delay.
before investment takes place. This occurs because, when cash flows rise exponentially rather than linearly, an increase in the growth rate has a greater impact on the option value to invest in the future than on the value of investing today.

A higher level of the interest rate implies a higher threshold level and a longer time until the investment takes place in models without a variable level of capital (e.g., [3, 4, 14]). However, an increase in the interest rate is found to lower the threshold level and hasten investment in Capozza and Li [6] where output is determined by the Cobb–Douglas production function. Here the interest rate effect is ambiguous, as we demonstrate in the next subsection.

3.5. Interest Rate Effects

The impact of interest rate changes on the threshold level and on investment is more complicated in this model. To transform the multiplicative relationship in (15) into an additive one, we study the interest rate elasticity of the threshold, i.e., the ratio of the percentage change in the threshold rent level, \( X^* \), to the percentage change in the level of the interest rate.

From (15), the interest rate elasticity of the threshold is

\[
\frac{\partial X^*}{\partial r} = \frac{1}{(1-\pi)(1-\delta)} \left( (1-\pi) - \frac{g}{r} \right),
\]

(16)

which is positive (negative) if \( g < (>) 1 - \pi \). This leads to the following proposition.

**Proposition 1.** Let cash flows, \( X \), increase at a constant rate, \( g \). Assume that the production function is CES with an elasticity of substitution, \( \pi \). Let \( CY \) be the current yield, \( IRR \) the internal rate of return, \( V \) the net present value, and \( q \) Tobin’s \( q \). Then for cash flows \( X \) near the threshold \( X^* \), if \( \frac{g}{r} \leq (>) 1 - \pi \),

\[
CY_r(X) \leq (>) CY^*_r, \quad IRR_r(X) \leq (>) IRR^*_r, \quad V_r(X) \leq (>) V^*_r, \quad q_r(X) \leq (>) q^*_r,
\]

where the partial derivatives with respect to the interest rate are positive in the first two inequalities and negative elsewhere.

The proof appears in the Appendix.

3.6. Discussion

The response of investment to interest rates is determined by the effect of the interest rate change on any of the decision variables relative to the effect on its critical value. For example, a positive response of investment to interest rate changes arises when an increase in the interest rate results in a larger increase in the IRR than in its critical value \( IRR^* \).\(^{13}\)

\(^{13}\)For the Cobb–Douglas production function (\( \pi = 1 \)), the positive effect always occurs. This result is consistent with the finding in Capozza and Li [6].
The positive effects of interest rate changes on the critical values $CY^*$ and $IRR^*$ are expected since they include the opportunity cost of capital. If the capital intensity is given exogenously, $CY$ and $IRR$ are independent of the interest rate. In this model, they are determined endogenously and hence vary with the interest rate. A higher interest rate reduces the optimal levels of capital and output. For production functions that exhibit diminishing returns to the variable factor, a lower level of output implies a higher output/capital ratio and a higher $CY$ and $IRR$.

As the interest rate increases, the present value of the cash flows decreases; and thus the NPV and Tobin’s $q$ decrease. An increase in the interest rate also reduces the present value of investing in the future and thus lowers the critical values for NPV and $q$.

Figures 2 and 3 illustrate the effect of a change in the interest rate on $IRR$, $IRR^*$, $NPV$, $V(X)$, $V^*$, and the value of the investment opportunity $W(X)$. Suppose that the interest rate rises from $r_1$ to $r_2$. Consider the case where $\frac{\pi}{r} < 1 - \pi$. In this case (illustrated in Fig. 2), the hurdle rental rate rises from $X_1^*$ to $X_2^*$. The increase in the interest rate raises $IRR$ and $IRR^*$ but lowers $V(X)$, $V^*$, and $W(X)$. Indeed, for projects with cash flows between $X_1^*$ to $X_2^*$, $IRR^*$ rises more than $IRR$ and the NPV falls more than the value of the investment opportunity. As a result, these projects that would have been undertaken are delayed.

Let us now turn to the case where $\frac{\pi}{r} > 1 - \pi$ (shown in Fig. 3). The hurdle rate falls from $X_1^*$ to $X_2^*$ as the interest rate rises from $r_1$ to $r_2$. While the increase in the interest rate still raises $IRR$ and $IRR^*$, but lowers $V(X)$ and $W(X)$, $IRR$ rises more than $IRR^*$ and $W(X)$ falls more than $V(X)$ for projects with cash flows between $X_2^*$ and $X_1^*$. Therefore, these projects, which would not have been undertaken when $IRR$ was less than $IRR^*$ or NPV was less than the value of the investment opportunity, are now acceptable since $IRR$ exceeds $IRR^*$ and NPV exceeds the value of the investment opportunity.

4. THE STOCHASTIC DECISION

We now turn to the stochastic case in which $\sigma > 0$ and expectations are rational. When the net rent evolves stochastically over time, the present value of rents and the value of the investment opportunity can be priced as contingent claims (options). To use the contingent claims approach, we assume that the stochastic changes in the state variable are spanned by existing assets in the economy. In this way, an equilibrium model such as the continuous-time version of the Capital Asset Pricing Model can be used to determine the risk-adjusted returns on the present value of cash flows and the option value to invest.

The present value of cash flows per unit in (2) is now given by

$$P(X) = \frac{X}{r - \hat{g}},$$

where $\hat{g}$ is the risk-neutral growth rate, $\hat{g} = g - \lambda$, and $\lambda$ is the risk premium for the portfolio replicating the stochastic evolution of the cash flow, $X$. 

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FIG. 2. The normal case: Shifts in NPV and IRR as the interest rate rises with low growth and volatility. \( V \) is NPV. \( W \) is the option value to invest. \( r \) is the interest rate. \( r_1 < r_2 \).

Since \( r - \hat{g} = (r + \lambda) - g \), the term \( r + \lambda \) is the risk-adjusted discount rate for cash flows. The option value to invest satisfies the following ordinary differential equation:\(^{14}\)

\[
\frac{\sigma^2}{2} X^2 W_{XX} + \hat{g} X W_X - r W = 0. \tag{18}
\]

\(^{14}\)Note that \( d \equiv r - \hat{g} \) is the divided yield, \( X/P \), from investing in an asset with price \( P \). \( \hat{g} = r - d \) does not reduce to the risk free rate since the dividend yield is not zero.
FIG. 3. The perverse case: Shifts in NPV and IRR as the interest rate rises with high growth and volatility. $V$ is NPV. $W$ is the option value. $r$ is the interest rate. $r_1 < r_2$.

The boundary conditions are

$$W(0) = 0, \quad W(X^*) = V(X^*), \quad W_X(X^*) = V_X(X^*). \quad (19)$$

The conditions have the usual interpretations. The first condition follows from the observation that the project is worthless if the current and future cash flows are zero. The second is the high-contact or continuity condition, stating that at the time of investment the value to invest in the future equals the project's
current NPV. The third is the “smooth-pasting condition” or first order condition, which ensures that the threshold, $X^*$, is chosen optimally.

The solution to the problem (18) subject to the first and second boundary conditions in (19) is (compare (6))

$$W(X) = V(X^*)\left(\frac{X}{X^*}\right)^\alpha = \left[\frac{Q(K^*)X^*}{r - \hat{g}} - K^*\right]\left[\frac{X}{X^*}\right]^\alpha,$$  \hspace{1cm} (20)

where

$$\alpha = \frac{1}{\sigma^2} \left[ -\left(\hat{g} - \frac{\sigma^2}{2}\right) + \sqrt{\left(\hat{g} - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2r} \right] > 1.$$  

Note that the deterministic or perfect foresight case is a special case of (20) since as $\sigma \to 0, \hat{g} \to g, \alpha \to r/g$. Also note that as $\sigma \to \infty, \alpha \to 1$.

4.1. Optimal Decision Rules in the Stochastic Case

The smooth-pasting condition in (19) is analogous to a first-order condition of (20) with respect to the threshold level $X^*$ (see Merton [15]). This condition implies the critical current yield,

$$\frac{Q^*X^*}{K^*} = r + \frac{\sigma^2}{2}.$$  \hspace{1cm} (21)

From (21), optimal investment occurs when the current yield equals the cost of capital, $r$, plus an uncertainty premium, $(\sigma^2/2)\alpha$.

From (9), the IRR of the project is the current yield plus the expected growth rate, $g$. Equation (21) implies that investment is made when the IRR reaches the hurdle rate given by

$$IRR^* = r + g + \frac{\sigma^2}{2}\alpha.$$  \hspace{1cm} (22)

Compared with the hurdle rate in the deterministic case given by (10), the hurdle rate, $IRR^*$, in the stochastic case contains an uncertainty premium since the value of waiting is higher under uncertainty. This hurdle rate differs from the required rate of return, $IRR^* = r + \lambda$, in the traditional theory, where $\lambda$ is the risk premium defined above.

The critical value of NPV is (from (21))

$$V^* = \frac{\hat{g} + (\sigma^2/2)\alpha}{r - \hat{g}} K^*$$  \hspace{1cm} (23)

which implies that, at the optimal time, the NPV yield, $V^*/K^*$, must equal the value of the growth, $\hat{g}/(r - \hat{g})$, plus the value of the volatility of the cash flows, $(\sigma^2/2)\alpha/(r - \hat{g})$.
From (21), the critical value for Tobin’s \( q \), is

\[
q^* = \frac{r + (\sigma^2/2)\alpha}{r - \hat{g}}.
\] (24)

The decision rules in the stochastic case are also summarized in Table 1. The investment policy for the case with stochastic cash flows is similar to that in the deterministic case (illustrated in Fig. 1) except that the optimal level of capital and the threshold rent level are higher.

The condition for the optimal level of capital is the same as (7). Dividing (7) by (21) and rearranging yields

\[
\gamma(K^*) = \frac{r - \hat{g}}{r + (\sigma^2/2)\alpha},
\] (25)

which is a stochastic generalization of (13).

4.2. An Explicit Solution

For the CES production function, the optimal level of capital and the threshold level of cash flows are

\[
K^* = \left( \frac{1/a - 1}{\alpha - 1} \right)^{1/\rho},
\] (26)

\[
X^* = \left( r + \frac{\sigma^2}{2} \alpha \right) \left( \frac{1 - a}{1 - 1/\alpha} \right)^{1/\rho}.
\] (27)

The comparative statics results with respect to the growth rate are the same as in the deterministic case. As in other option models, the option value to invest increases in the volatility of cash flows. Thus, the higher the volatility is, the longer the time until the investment is undertaken. In addition, the higher the volatility is, the higher the level of capital that will be committed when the investment takes place.

4.3. Interest Rate Effects

The interest rate elasticity of the threshold level is

\[
\frac{\partial X^*}{\partial r} \frac{r}{X^*} = \frac{1}{(1 - \pi)(1 - \frac{\hat{g}}{r})} (1 - \pi - \eta),
\] (28)

where

\[
\eta = \frac{(r - \hat{g})}{a(\alpha - 1)\sqrt{(\tilde{g} - \sigma^2/2)^2 + 2\sigma^2r}}.
\]

which is increasing in \( \hat{g} \) and \( \sigma \) and approaches \( g/r \) as \( \sigma \to 0 \) and one as \( \sigma \to \infty \). Thus, for any \( g \), if the volatility \( \sigma \) is sufficiently large, then \( \eta > 1 - \pi \), and the interest rate effect is negative: \( (\partial X^*/\partial r)(r/X^*) < 0 \).
Proposition 2. Assume that cash flows, $X$, evolve according to a geometric Brownian motion with drift $g$ and standard deviation $\sigma$. Also assume that the production function is CES with elasticity of substitution $\pi$. Let $CY$ be the current yield, $IRR$ the internal rate of return, $V$ the net present value, and $q$ Tobin’s $q$. Then for cash flows $X$ near the threshold $X^*$, if $\eta \leq (1 - \pi)$, 

$$
CY_r(X) \leq (>)CY^*_r, \quad IRR_r(X) \leq (>)IRR^*_r,
$$

$$
V_r(X) \leq (>)V^*_r, \quad q_r(X) \leq (>)q^*_r,
$$

where the partial derivatives with respect to the interest rate are positive in the first two inequalities and negative elsewhere.

The proof is also given in the Appendix.

4.4. Discussion

As in the deterministic case, an increase in the interest rate accelerates development decisions if the interest rate effect on the project (the left-side arguments) is greater than the effect on the hurdle (the right-side arguments). An increase in the interest rate raises the $IRR$ and $IRR^*$ of the project, but lowers the $NPV$ and option value to invest. If volatility is low, $IRR^*$ rises more than $IRR$ and the $NPV$ falls more than the option value to invest. Thus, some projects that would have been undertaken are delayed as the interest rate rises. However, if volatility is high, $IRR$ rises more than $IRR^*$ and the option value falls more than the $NPV$. Consequently, investments that would not have been made are hastened (see Figs. 2 and 3).

Note that the hurdle rate $IRR^*$ and the option value of the project are positively related to the volatility. If the volatility is high, there is more delay before investment is made. The positive interest rate effect arises when projects awaiting development are undertaken when the interest rate rises.

Figure 4 summarizes the effect of a change in the interest rate on the timing of investment for various levels of the expected growth rate and volatility. Let $(\bar{g}, \bar{\sigma})$ be any point located on the interest-rate-neutral curve, i.e., the collection of $(g, \sigma)$ at which the interest rate changes have no impact on investment. For a given level of the interest rate and a given level of the elasticity of substitution, the positive effect of interest rate changes on investment arises if $g > \bar{g}$ or $\sigma > \bar{\sigma}$. The interest-rate-neutral curve shifts downward as the elasticity of substitution rises. This implies that the larger the elasticity of substitution is, the more likely the positive effect.

4.5. Interest Rate Effects in Other Real Options Models

A number of authors have discussed the possibility of positive interest rate responses in real options models. The possibility was first raised by Heaney and Jones [11]. In their model, the interest rate response balances the negative effect of interest rates on the present value of the cash flows with the negative effect
on the value of waiting. For short-lived projects the first effect on the PV of the cash flows is small so that the second effect on the value of waiting dominates. Capozza and Li [5] find positive responses in an urban context with infinitely lived projects. In their model the exercise price includes the opportunity cost of the land, which falls when interest rates rise. The positive interest rate effect arises when the opportunity cost of the land is high relative to the capital needed for the project and uncertainty is high. Amin and Capozza [1] develop a two-factor model and find that the Capozza and Li [5] result holds when both interest rates and rents are stochastic. Ingersoll and Ross [12] develop a model of real investment that focuses on stochastic interest rates and bullet projects where returns are concentrated in a single time period. In their model, as in Heaney and Jones, the positive effect occurs for short duration projects. The real options model of Capozza and Li [6] where production technology is Cobb–Douglas, an extreme case of the CES technology, always exhibits the positive response.
to interest rates. That is, the inequalities in parentheses in Propositions 1 and 2 always hold.

5. CONCLUSIONS

In this paper we have analyzed land development decisions with variable capital intensity. Two types of theoretical results are derived. First, simple decision rules for optimal investment timing are derived and elaborated in terms of current yield, internal rate of return, net present value yield, and $q$ ratio. In a growing economy the simplest optimal timing rule uses the current yield. Optimal investment occurs when the current yield equals the cost of capital in the deterministic case and equals the cost of capital plus an uncertainty premium $(r + (\sigma^2 \alpha/2))$ in the stochastic case. The IRR equals the cost of capital plus the growth rate of cash flows $(r + g)$ under certainty. With uncertainty a premium must be included $(r + g + (\sigma^2 \alpha/2))$. The net present value and $q$ rules are only slightly more complicated (see Table 1). Note that investments are optimally delayed relative to traditional investment criteria even under certainty if cash flows are growing.

Second, we derive the conditions under which positive responses of investment to interest rate increases can occur. In the model uncertainty is not a necessary condition for these positive responses. High growth rates are sufficient to cause the positive response. High volatility increases the likelihood of positive responses when growth rates are positive and is sufficient if growth is zero. Intuitively, an increase in the interest rate raises both the hurdle IRR and the project IRR. The project IRR rises because higher interest rates reduce the optimal capital intensity and increase the corresponding output/capital ratio. If the project IRR rises more than the hurdle IRR, projects are accelerated in time rather than delayed by the interest rate increase. When the elasticity of substitution between capital and other factors is high, acceleration is more likely.

The model could be extended in a number of ways. First, we have assumed that output is sold in a competitive market so that there is no interaction between the capital intensity and the price of output. For investments in monopolistic settings, demand will be downward sloping. A less than perfectly elastic demand should weaken the effect of capital intensity on investment decisions. Second, as with most models of real investment decisions, our model is a one-stochastic-factor model. We have modeled only the process for project revenue. As indicated above, if the process for long-term interest rates is stochastic and it has a temporary as well as a permanent component, the optimal reactions to interest rate changes will be smaller. We conjecture that this should not, however, affect the sign of the optimal reaction.$^{15}$

$^{15}$Amin and Capozza [1] provide such a model with stochastic interest rates. Analytical results cannot be obtained, but qualitative results are similar to the one-stochastic-factor model.
The policy implications of these results are profound. If increases in the interest rate can accelerate investment spending then extreme care must be taken when monetary decisions are designed to transmit restraint to the economy through their effect on investment. It becomes extremely important to understand the conditions under which the perverse results are obtained. Since it is the ratio of the growth rate to the interest rate that is critical, positive interest rate responses can occur either when growth rates are high or when interest rates are very low.

There are also regional implications to the results. Since some parts of the country grow faster than others, interest rate policy can have very different effects regionally. Indeed, the regions most in need of restraint from interest rate policy may also be the areas most likely to respond perversely.

**APPENDIX**

*Proof of Proposition 2.* (The Proof of Proposition 1 is a special case in which \( \sigma \to 0 \).) We first prove the inequality for \( CY \). Since \( CY(X) \) is a continuous function of \( X \), it is sufficient to consider the case in which \( X = X^* \).

In this case, \( CY(X^*) = CY^* \) and thus \( CY(X^*) < (>) CY^* \) is equivalent to \( cYi < (>) cY^i \) where \( cY = \log(CY) \) and \( l = \log(r) \). Since \( cy - cY^i = -x^i \), the first inequality follows from the equation before the proposition.

Note that alpha is the root of the quadratic equation \((\sigma^2/2)\alpha(\alpha - 1) + \hat{g}\alpha - r = 0\). The right-hand side of (21) can also be written as \( \varphi r \) where \( \varphi = (1 - \hat{g}/r)/(1 - \alpha) > 1 \), which approaches unity as \( \sigma \to 0 \). To show \( CY_r > 0 \), note that \( CY_r = -[(\varphi - 1) + (\hat{g}/r)]k^i_*/\varphi > 0 \) where \( \varphi > 1 \) and \( k^i_* = -\varphi/[\rho(\alpha - 1)] < 0 \). To show \( CY^*_r > 0 \), note that \( CY^*_r = 1 + \varphi_r/\varphi = (1 - \eta)/(1 - \hat{g}/r) > 0 \) where \( \eta < 1 \).

Note that \( IRR(X) = CY(X) + g, V(X) + K^* = CY(X)K^*/(r - \hat{g}) \), and \( q(X) = CY(X)/(r - \hat{g}) \). Thus

\[
\log[IRR(X) - g] - \log(IRR^* - g) = \log[V(X) + K^*] - \log(V^* + K^*) = \log[q(X)] - \log(q^*) = \log[CY(X)] - \log(CY^*).
\]

The inequalities for \( IRR, V, \) and Tobin’s \( q \) can be shown analogously.

**REFERENCES**