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Undiscounted optimal growth with consumable capital and labor-intensive consumption goods ☆

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A B S T R A C T

This paper presents a complete characterization of the optimal policy in a two sector undiscounted growth model. The model is an extension of the Leontief two sector model, which analyzes the optimal allocation of capital and labor to a consumption good sector and an investment good sector. The paper extends this framework to include consumable capital. Thus, the planner has preferences over the consumption good and the consumable capital. Future welfare levels are treated equally as current ones. Geometric techniques are applied to characterize the optimal policy if the consumption good is labor-intensive. The results suggest that if the initial capital stock is above a threshold level, that depends upon the consumption of capital, every optimal program is monotonic, converges to the golden rule stock in a finite number of periods, and undergoes either unemployment or excess capacity of capital.

1. Introduction

Ramsey (1928) addressed the problem of undiscounted optimal growth in which the optimal program of capital accumulation was derived from the maximization of a utility sum over an infinite time period. Samuelson and Solow (1956) extended the framework to a model with many capital goods. Atsumi (1965) and Von Weizsacker (1965) proposed the overtaking criterion approach. Under this criterion, Gale (1967) and Brock (1970) showed the existence of the optimal path. Approaching a special case of the two sector model proposed by Nishimura and Yano (1995, 1996, 2000), Fujio (2005, 2008, 2009) employed geometric techniques developed by Khan and Mitra (2007) to characterize the optimal policy.

This paper uses a variant of the Leontief two sector growth model which analyzes the optimal allocation of capital and labor to a consumption sector and an investment sector in every period. This paper, however, attempts to extend the previous analysis in Fujio (2005, 2008, 2009) to the case of consumable capital; that is both the consumption good and capital can be consumed. Accordingly, a planner has preferences defined over both the consumption good and consumable capital. Future welfare levels are treated equally as current ones; that is the discount factor is assumed unity. The utility function with two arguments: the consumption good and the consumable capital, has implications on the planner's preferences and consequently on the value-loss and the optimal program.

The paper uses geometric techniques developed by Khan and Mitra (2007) to characterize the optimal program. The optimal program depends on the factor intensity of the consumption good and on parameter values; principally the marginal rate of transformation of capital between today and tomorrow. This paper focuses on the case of the labor-intensive consumption good. Khalifa (2010) considers the case of the capital-intensive consumption good.

When the consumption good is labor-intensive, every optimal program monotonically converges to the golden rule stock within a finite number of periods, and undergoes either unemployment or excess capacity of capital. Countries with a low initial capital stock produce investment goods only, while countries with a high initial capital stock produce consumption goods only. In this context, a capital poor country must overcome a period of no consumption during the beginning of a program, a capital rich country experiences a no investment phase during the beginning of a program, and finally countries in the middle range reach the golden rule stock within the first period. This optimal policy, however, departs not only from the case of the capital-intensive consumption good with consumable capital, considered in Khalifa (2010), but also from the case of the labor-intensive consumption good without consumable capital, considered in Fujio (2005). In this context, the policy implications hold only if the initial capital stock is above a certain threshold, which depends on the amount of the consumption of capital. Otherwise, the economy will not converge to the golden rule stock.

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This implies that the characterization of the optimal program has policy implications, as the model specifies the program to be adopted to ensure the sustainability of resources. An example in this context is water resources. Water can be used in irrigation to produce an agricultural consumption good, and can also be used directly for drinking purposes. Therefore, water can serve as capital that is used in the production process of a consumption good, and can be consumable as well. This specific application is pertinent to developing countries that has a significant agricultural and irrigation sector. These countries are faced with the challenge of the scarcity of water resources, and the tradeoff between using these resources for consumption and productive purposes.

The remainder of the paper is organized as follows: Section 2 presents the model, and Section 3 includes the concluding remarks. References and graphs are included thereafter.

2. Model

2.1. Technology and preferences

Consider the two sector model that consists of a consumption good sector and an investment good sector. Each sector uses two factors of production: labor and capital. The amount of labor is normalized to unity in every period of time; that is the growth rate of labor is assumed zero. The amount of capital available at the beginning of period 0 is exogenously given and denoted $x(0)$. At any moment in time, a planner allocates an amount of capital and labor to either sector.

For each date $t\in\mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers, we assume that each sector employs a Leontief type technology for each date t. The production of one unit of the consumption good requires one unit of labor and a_c units of capital. The production of b units of the investment good requires one unit of labor and a_I units of capital. Let the technologies be stationary and given by

$$c(t+1) = min\left\{l_c(t), \frac{1}{a_c}k_c(t)\right\} \tag{1}$$

$$z(t+1) = b\ min\left\{l_I(t), \frac{1}{a_I}k_I(t)\right\} \tag{2}$$

where $c(t+1)$ and $z(t+1)$ denote the amounts of output of the consumption good sector and the investment good sector at $t+1$, respectively. For $i\in(C,I)$, $l_i(t)$ and $k_i(t)$ denote the amount of labor and capital inputs in each sector at period t, respectively. We assume that capital can be consumed. The amount of capital that is consumed at period t is given by $w(t)$. We also assume that the stock of capital depreciates at a rate $d\in(0,1)$ each period. The residual stock plus the investment goods produced in the same period form next period's capital stock. If $x(t)\geqslant 0$ denotes capital stock available in period t, then

$$x(t+1) = (1-d)x(t) + z(t+1). \tag{3}$$

As labor is normalized to unity in every period of time, we have the labor constraint $0<l_c(t)+l_I(t)\le 1$, while the capital constraint can be written as $0<k_c(t)+k_I(t)\le x(t)$. Substituting $c(t+1)\le l_c(t)$, and $\frac{z(t+1)}{b}\le l_I(t)$ give us the following labor constraint

$$0\le c(t+1)\le 1-\frac{x(t+1)-(1-d)x(t)}{b}. \tag{4}$$

This is since the amount of labor required to produce $z(t+1)$ of the investment good is $\frac{z(t+1)}{b} = \frac{x(t+1)-(1-d)x(t)}{b}$. Since labor is normalized to unity, the labor left to produce the consumption good is $1-\frac{x(t+1)-(1-d)x(t)}{b}$. Similarly, substituting $a_c c(t+1)\le k_c(t)$, and $\frac{a_I z(t+1)}{b}\le k_I(t)$ give us the following capital constraint

$$0\le c(t+1)\le\frac{1}{a_c}\left[x(t)-\left(\frac{a_I}{b}\right)(x(t+1)-(1-d)x(t))\right]. \tag{5}$$

This is since the capital required to produce $z(t+1)$ of the investment good is $\frac{a_I z(t+1)}{b}$. Thus, the capital left to produce the consumption good is $x(t)-\frac{a_I z(t+1)}{b}$. This amount of capital produces $\frac{1}{a_c}\left[x(t)-\frac{a_I z(t+1)}{b}\right]$ of the consumption good.

Definition 1. *A program from $x_0\in\mathbb{R}_+$ is a sequence $\{x(t),w(t)\}$ with $(x(t), w(t))\in\mathbb{R}_+\times\mathbb{R}_+$ such that $x(0)=x_0$, and for all $t\in\mathbb{N}$*

$$x(t+1)\ge(1-d)x(t)$$
$$0<k_c(t)+k_I(t)\le x(t)$$
$$0<l_c(t)+l_I(t)\le 1.$$

A program $\{x(t),w(t)\}$ is a program from $x(0)$.

Definition 2. *Associated with any program $\{x(t),w(t)\}$ is a gross stock increase sequence $\{z(t+1)\}$ with $z(t+1)\in\mathbb{R}_+$, and a consumption sequence $\{c(t+1)\}$ with $c(t+1)\in\mathbb{R}_+$, such that for all $t\in\mathbb{N}$*

$$z(t+1) = x(t+1)-(1-d)x(t)$$
$$c(t+1) = min\left\{l_c(t), \frac{1}{a_c}k_c(t)\right\}.$$

Definition 3. *A program $\{x(t),w(t)\}$ is called stationary if for all $t\in\mathbb{N}$*

$$(x(t),w(t)) = (x(t+1),w(t+1)).$$

The preferences of the planner are represented by a felicity function $\upsilon:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$. We assume that the felicity function $\upsilon(c(t), w(t))$ is separable in its two arguments: the consumption good and the consumable capital good. Future welfare levels are treated equally as current ones; that is the discount factor is assumed unity. We impose the following assumptions on the felicity function with respect to its two arguments: (1) the marginal utility of the consumption good is constant, (2) the marginal utility of consumable capital is larger than the marginal utility of the consumption good up to a threshold $w(t)=A$, and (3) the marginal utility of consumable capital is negative after the threshold $w(t)=A$. These assumptions ensure that if the initial capital stock is less than the threshold A, the planner prefers to use all the available stock of capital for consumption, and none for the production of either the consumption good or the investment good. While if the initial capital stock is higher than the threshold, the planner prefers to consume an amount that exactly equals the threshold A only.

Khalifa and Hurcan (forthcoming) suggest water resources as an example of consumable capital that can justify these assumptions. Water can be used in irrigation to produce an agricultural consumption good, and can also be used directly for drinking purposes. Therefore, water can serve as capital that is used in the production process of a consumption good, and can be consumable as well. A dataset from the World Resources Institute[1] is used to show the percentage of total water withdrawals used for domestic purposes including water consumption, and that used for agricultural and industrial purposes. Fig. 7 depicts the relationship between total water withdrawals in countries around the world, and the percentage of these withdrawals dedicated to domestic purposes or water consumption. The figure shows the actual percentages corresponding to every level of total water withdrawals, in addition to the trend. The declining trend can provide a justification for our assumption that if

[1] Detailed data description is included in Appendix A.

the available stock of water is lower than a threshold A, the planner prefers to use all the available stock for drinking and accordingly the percentage that is used for water consumption is high. However, as the available stock gets larger than the threshold, the planner prefers to use the exact amount A for drinking purposes. This means that as the stock gets larger, the percentage dedicated for water consumption gets smaller, consistently with the observed trend.

Following Khan and Mitra (2005), we work with the overtaking criterion of optimality.

Definition 4. *A program* $\{x^*(t), w^*(t)\}_{t=0}^{\infty}$ *starting from* $x(0)$ *is called optimal if there does not exist any* $\varepsilon>0$ *and a time period* t_ε *such that*

$$\sum_{t=1}^{T}\left[v(c(t),w(t))-v\left(c^*(t),w^*(t)\right)\right]\geq \varepsilon \text{ for all } T\geq t_\varepsilon.$$

That is an optimal program is one in comparison to which no other program, for the same initial stock, is eventually significantly better for any given level of significance. A program is said to be a stationary optimal program if it is stationary and optimal.

2.2. Reduced form

Following McKenzie (1968), we convert the model into its reduced form which is summarized by the transition possibility set Ω as a collection of pairs (x,x'), such that it is possible to have x' of the amounts of the capital stock in the next period from the x amounts of capital stock available in the current period. Formally,

$$\Omega=\left\{(x,x')\in\mathbb{R}_+\times\mathbb{R}_+:(1-d)x\leq x'\leq b\ \min\left\{1,\frac{x}{a_I}\right\}+(1-d)x\right\}. \quad (6)$$

This is since next period's stock is at least the left over after depreciation. Otherwise, a_I units of capital are used to produce b units of the investment good, or x units are used to produce $\frac{bx}{a_I}$ units. Using this formulation, we can keep track of the transition dynamics of only the state variable, $x(t)$, overtime. For any $(x,x')\in\Omega$, one can consider the amounts w. Formally, we have a correspondence $\Psi:\Omega\to\mathbb{R}_+\times\mathbb{R}_+$, given by

$$\Psi(x,x')=\left\{\begin{array}{c} w\in\mathbb{R}_+:0\leq c\leq 1-\left(\frac{1}{b}\right)[x'-(1-d)x] \\ \text{and } 0\leq c\leq\frac{1}{a_c+A}\left[x-\left(\frac{a_I+A}{b}\right)(x'-(1-d)x)\right]\end{array}\right\}. \quad (7)$$

Finally, the reduced form utility function u: $\Omega\to\mathbb{R}_+$, is defined on Ω such that

$$u(x,x')=\max\ \{v(c,w):(c,w)\in\Psi(x,x')\}. \quad (8)$$

2.3. Geometry

The case of the labor-intensive consumption good, where $a_I>a_C$ is shown in Fig. 1. Khalifa (2010) considers the case of the capital-intensive consumption good. The x-axis stands for the amount of capital stock available today, while the x'-axis stands for the amount of capital stock available tomorrow. To proceed with the geometric analysis, it is assumed that $b>d(a_I+A)$ in order to guarantee the existence of the golden rule stock.

The 45° line plays an important role to track a program period by period. The enclosed region by PV, VN, and PL is the transition possibility set, Ω, which gives a possible range of x' for every x. We call a particular point in Ω a plan. A plan on the straight line PL stands for specialization in the production of consumption goods. If the initial

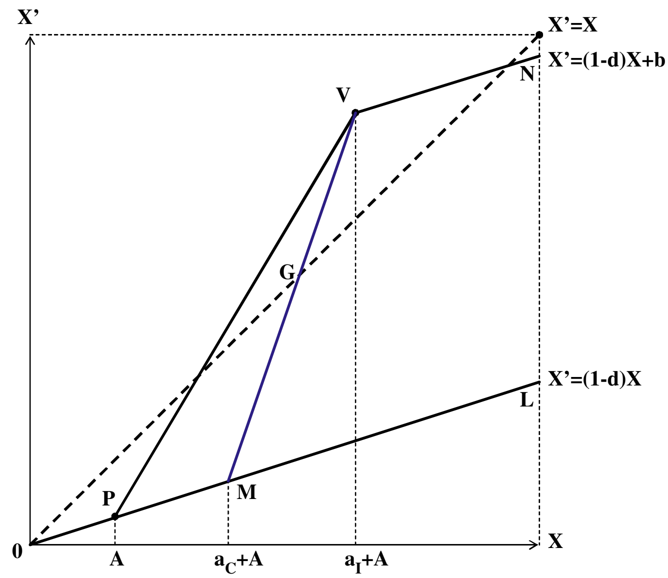

Fig. 1. The labor-intensive consumption good case.

capital stock $x\leq A$, it is consumed entirely and we are left with $(1-d)x$. If $x>A$, an amount A is consumed, and the remainder $(x-A)$ is used to produce the consumption good. Since there is no production of the investment good, we are left with $x'=(1-d)x$. A plan on the line PVN which has a kink at $x=a_I+A$, stands for specialization in the production of investment goods. At any interior point of Ω, both consumption and investment goods are produced.

In this context, the line VM plays a central role in our geometric investigation. It is referred to as the "full employment-no excess capacity line", as it provides us with the set of plans with fully utilized labor and capital. At $x=a_I+A$, the planner consumes A units of capital, and the remainder a_I is used with all the available labor, that is normalized to unity, to produce b units of the investment good. In this case, $x'=(1-d)(a_I+A)+b$. At $x=a_C+A$, the planner consumes A units of capital and uses the remaining a_C with all the available labor to produce one unit of the consumption good. Nothing is left for investment, and thus the planner ends up with $x'=(1-d)(a_C+A)$ only.

In Fig. 2, it is obvious that the slope of the full employment-no excess capacity line VM is given by

$$\frac{VQ'}{Q'M}=\frac{(1-d)(a_I+A)+b-(1-d)(a_C+A)}{(a_I+A)-(a_C+A)}=\frac{b}{a_I-a_C}+(1-d)\equiv\xi. \quad (9)$$

To obtain the equation of the line VM, we pick any point, say $S=(x,x')$ as in Fig. 2. Since $\Delta MVQ'$ and $\Delta VSS'$ are similar triangles, we have

$$\frac{b+(1-d)(a_I+A)-x'}{b+(1-d)(a_I+A)-(1-d)(a_C+A)}=\frac{VS'}{VQ'}=\frac{SS'}{MQ'}=\frac{(a_I+A)-x}{(a_I+A)-(a_C+A)}. \quad (10)$$

Therefore, the equation for the line VM is given by

$$x'=b\left(\frac{x-A-a_C}{a_I-a_C}\right)+(1-d)x=-\xi x+\frac{b(a_C+A)}{ac-a_I}. \quad (11)$$

Similarly, the line VM separates the transition set into two parts: PVM and $LMVN$. In those regions, excluding the line VM, there exists either not utilized labor or excess capacity of capital. Consider any plan on PVM and not on VM, say $S_1=(x,s_1)$ in Fig. 2, at which the amount of production of the investment good is higher than the full utilization level by (s_1-x').

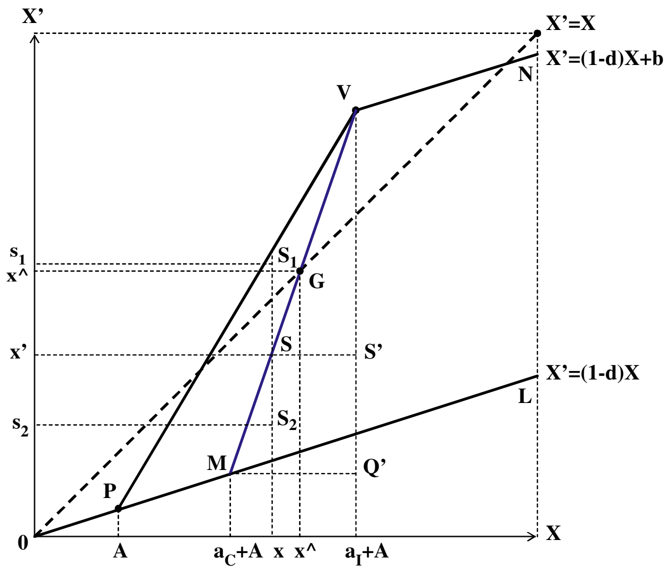

Fig. 2. The slope of the full employment-no excess capacity line when $a_I > a_C$.

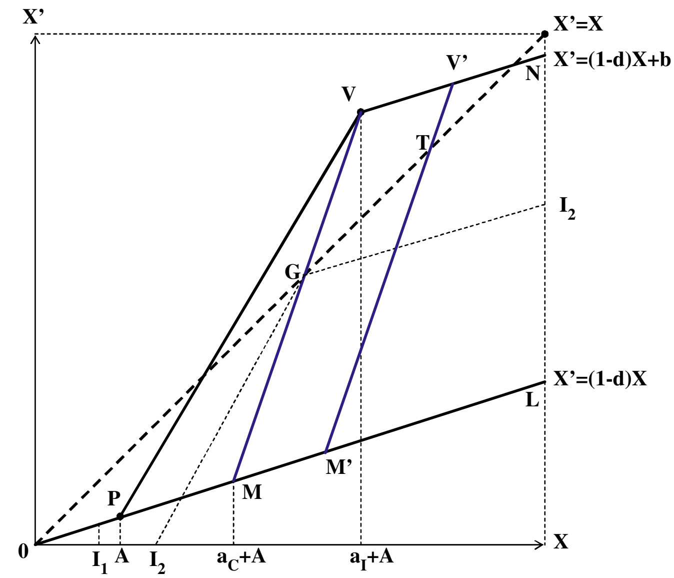

Fig. 3. The planner's preference map.

The amount of labor and capital employed in the investment good sector at S_1 is higher than that at the full employment level S by $\left(\frac{s_1 - x'}{b}\right)$ and $(s_1 - x')\left(\frac{a_I}{b}\right)$, respectively. The consumption sector releases $(s_1 - x')\left(\frac{a_I}{b}\right)$ capital, which is all used in the investment sector. The consumption sector releases $\left(\frac{s_1 - x'}{b}\right)\left(\frac{a_I}{ba_C}\right)$ labor, but only $(s_1 - x')\left(\frac{a_I}{b}\right)$ is employed in the invesment sector and thus the difference $\left(\frac{s_1 - x'}{b}\right)\left(\frac{a_I}{ba_C}\right) - \left(\frac{s_1 - x'}{b}\right) = \left(\frac{s_1 - x'}{b}\right)\left(\frac{a_I - a_C}{ba_C}\right)$ is unemployment.

Next, consider a plan in $LMVN$ and not on VM, say $S_2 = (x, s_2)$ at which the amount of production of the investment good is lower than the full utilization level by $(x' - s_2)$. The amount of labor and capital employed in the investment good sector at S_2 is lower than that at the full employment level S by $\left(\frac{x' - s_2}{b}\right)$ and $(x' - s_2)\left(\frac{a_I}{b}\right)$, respectively. The investment sector releases $\left(\frac{x' - s_2}{b}\right)$ labor, which is all used in the consumption sector. The investment sector also releases $(x' - s_2)\left(\frac{a_I}{b}\right)$ capital, but only $\left(\frac{x' - s_2}{b}\right)a_C$ are used in the consumption sector and thus the difference $(x' - s_2)\left(\frac{a_I}{b}\right) - \left(\frac{x' - s_2}{b}\right)a_C = (x' - s_2)\left(\frac{a_I - a_C}{b}\right)$ is the amount of excess capacity.

If $x(t) \in (0, A]$, the consumption of capital $w(t) = x(t)$, with a zero level of the consumption good. Therefore, the consumption good is not influencing the utility of the planner in this segment which is solely determined by the amount of consumable capital. On the other hand, if $x \geq A$, the consumption of capital is constant at A. Therefore, consumable capital is not influencing the utility of the planner in this segment which is solely determined by the amount of the consumption good. Therefore, the planner's indifference map over the (x, x')-plane can be easily configured. The indifference map of the planner if $x \in (0, A]$ is a set of vertical lines parallel to I_1 as in Fig. 3. The reason is that, at this segment, the level of consumable capital in any plan in an indifference curve is exactly equal to the available capital x while the level of the consumption good equals to zero. In addition, any horizontal movement to the right of a plan on an indifference curve is accompanied by a higher level of consumable capital and accordingly a higher level of utility.

The indifference curves are, however, kinked on the VM line as shown in Fig. 3. Consider the line I_2 on the VM line. By specification, the amounts of labor and capital used in the consumption good sector are all identical to the amount of the consumption good. If we move vertically downwards on I_2 from the highest x', this implies a decrease in the terminal stock of capital x' and a consequent increase in the labor available to the consumption good sector. However, this sector is constrained by the available capital and hence no increase in the consumption good and thereby in utility is possible. On the other hand, consider a horizontal move from the highest x' on I_2 in a direction parallel to PL. This move represents an increase in the initial capital stock x and a consequent increase in the capital available to the consumption good sector. As no labor is being released, this sector faces a labor constraint. Hence, no increase in either the consumption good or in utility is possible. Obviously, because the planner chooses to consume an exact amount A of capital, it does not affect our indifference curves. Thus, we obtain an indifference curve I_2 of the Leontief type with a kink at its intersection with VM. If the initial capital stock increased and the terminal stock decreased, relative to I_2, an increased amount of both capital and labor is made available to the consumption sector, resulting in a higher level of the consumption good and thereby utility. The line VM thus takes on a new identity; it pegs a map of kinked indifference curves in the (x, x')-plane.

2.4. Benchmarks

We first derive a golden rule capital stock, $\hat{x}$, and a golden rule price level, $\hat{p}$, and then we define a value-loss for every single plan $(x, x') \in \Omega$ in terms of $\hat{x}$ and $\hat{p}$. Second, we show that the VM line is the zero-value-loss line and that lines parallel to VM are iso-value-loss lines. Finally, we show that a value-loss of a single plan is measured by the amount of vertical deviation from the plan on the VM line.

The golden rule stock is defined as a point of maximal sustainable utility, where the terminal capital stock must be as large as the initial. The level of the golden rule stock, at which the maximal utility is sustainable every period, is derived as a solution to the problem

$$\max\ u(x, x') \ \text{ subject to } \ x' \geqslant x \ \text{ for all } \ (x, x') \in \Omega. \tag{12}$$

Fig. 2 shows the golden rule stock to be the unique one period plan $G = (\hat{x}, \hat{x})$ obtained by the intersection of the 45° line with VM. In this case, $\hat{x}$ can be derived as the solution to $\hat{x} = -\xi\hat{x} + \frac{b(a_C + A)}{a_C - a_I}$ which yields

$$\hat{x} = \frac{b(a_C + A)}{b + d(a_C - a_I)}. \tag{13}$$

Since $(\hat{x},\hat{x})$ is the solution to the problem, appealing to Uzawa's version of the Kuhn –Tucker theorem, there exists a golden rule price level $\hat{p}$ such that

$$u(x,x') + \hat{p}(x'-x) \leq u(\hat{x},\hat{x}) \text{ for all } (x,x') \in \Omega. \quad (14)$$

Proposition 1. *The golden rule price system* $\hat{p} = \frac{\hat{x}}{b(a_C + A)}$.

Proof. Included in Appendix A. □

We can define a value-loss such that

$$\delta_{(\hat{p},\hat{x})}(x,x') = u(\hat{x},\hat{x}) - u(x,x') - \hat{p}(x'-x) \text{ for all } (x,x') \in \Omega. \quad (15)$$

Proposition 2. *The VM line is the zero-value-loss line, that is* $\delta_{(\hat{p},\hat{x})}(x,x') = 0$ *for any* (x,x') *such that* $x' = -\xi x + \frac{b(a_C + A)}{a_C - a_I}$.

Proof. Included in Appendix A. □

Proposition 3. *A line parallel to VM, say V′M′, is an iso-value-loss line, and the value-loss of a plan on V′M′ can be measured by the difference of the golden rule utility level and the utility level of a plan at which V′M′ and the 45° line intersects.*

Proof. Included in Appendix A. □

Proposition 4. *For any plan* $(x,x') \in \Omega$*, the more a plan is vertically deviated from the plan on the zero-value-loss line VM the more value-loss it suffers.*

Proof. Included in Appendix A. □

Using this fact, we can derive a conclusion on the aggregate value-loss of several plans.

Proposition 5. *The sum of the value-losses of two plans equals the value-loss of the sum of two plans.*

Proof. Included in Appendix A. □

2.5. Optimal policy

In this section, we present a complete characterization of optimal programs. An optimal program is one that minimizes the aggregate value-loss and converges to the golden rule stock. Alternatively, any program that minimizes the aggregate of the sequence of all value-losses over the long run is an optimal trajectory. Using cob-web diagrams in today-tomorrow plane, every program starting from any initial capital stock can be tracked period by period, and its associated value-loss per period and then its aggregate value-loss can be calculated. In this way, we can compare the aggregate value-losses of two different programs starting from the same initial capital stock and find an optimal program which has the minimum aggregate value-loss. The line *VM* is referred to as the von-Neumann facet, the golden rule stock is the von-Neumann point, and the McKenzie facet is defined as the part of the von-Neumann facet which has the property that any optimal program starting from it remains on it.

There are several cases to consider depending on the value of ξ. This paper focuses on the case of $\xi < -1$, while Khalifa (2010) considers all other cases. If the consumption good is labor-intensive, it means that $a_C < a_I$ and $\xi < -1$. This case is shown in Fig. 5. For any initial capital stock $x(0) \in [a_C + A, a_I + A]$ a program taking a plan on *VM* gets the level of capital stock in the next period even further away from the golden rule stock. Eventually the program gets out from the von-Neumann facet and starts to accumulate value-losses without bound. Unless it starts from the von-Neumann point, an optimal program does not remain on the von-Neumann facet.

Proposition 6. *If* $\xi < -1$*, and if* $b < d(a_C - a_I)$*, the McKenzie facet shrinks to the von-Neumann point.*

Proof. Included in Appendix A. □

Since every good program converges to the golden rule capital stock, we need to find a program which gets to *G* at some point with a minimum value-loss. As long as a program does not start from the von-Neumann point, it has to suffer unemployment or excess capacity of capital at some point.

Proposition 7. *If* $\xi < -1$*, the optimal policy is given by*

$$x(t+1) = \begin{cases} \left(\frac{b}{a_I} + 1 - d\right)x(t) - \frac{b}{a_I}A & \text{for } x^* < x(t) < \delta \\ \hat{x} & \text{for } \delta \leq x(t) \leq \frac{\hat{x}}{(1-d)} \\ (1-d)x(t) & \text{for } x(t) > \frac{\hat{x}}{(1-d)} \end{cases}$$

where $\delta = \frac{\hat{x} + \frac{b}{a_I}A}{\left(\frac{b}{a_I} + 1, -d\right)}$.

Proof. Included in Appendix A. □

The case when the initial capital stock is below the threshold x^* is considered in the following proposition.

Proposition 8. *If* $\xi < -1$*, and* $x \leq x^* = \frac{\frac{b}{a_I}A}{\left(\frac{b}{a_I} - d\right)}$*, there is no convergence to the golden rule stock.*

Proof. Included in Appendix A. □

3. Conclusion

This paper uses a variant of the Leontief two sector undiscounted optimal growth model which analyzes the optimal allocation of capital and labor to a consumption sector and an investment sector in every period. The paper, however, attempts to extend the previous analysis to the case of consumable capital; that is both the consumption good and capital can be consumed. Accordingly, a planner has preferences defined over both the consumption good and consumable capital. Future welfare levels are treated equally as current ones, that is the discount factor is assumed unity. The utility function with two arguments: the consumption good and the consumable capital, has implications on the planner's preferences, on the value-loss and accordingly on the optimal program. The paper uses geometric techniques developed by Khan and Mitra (2007) to characterize the optimal program.

When the consumption good is labor-intensive, every optimal program monotonically converges to the golden rule stock within a finite number of periods, and undergoes either unemployment or excess capacity of capital. Countries with a low initial capital stock produce investment goods only, while countries with a high initial capital stock produce consumption goods only. In this context, a capital poor country must overcome a period of no consumption during the beginning of a program, a capital rich country experiences a no investment phase during the beginning of a program, and finally countries in the middle range reach the golden rule stock within the first period. However, these results hold only if the initial capital stock is above a certain threshold, which depends on the amount of the consumption of capital. Otherwise, the economy will not converge to the golden rule stock.

Appendix A

Proof of Proposition 1. Appealing to Uzawa's version of the Kuhn–Tucker theorem, there exists a golden rule price level $\hat{p}$ such that

$$u(\hat{x},\hat{x})-u(x,x')\geq\hat{p}\,(x'-x).$$

Substituting the zero consumption plan $V=(a_I+A,(1-d)(a_I+A)+b)$, and

$$u(\hat{x},\hat{x})=\frac{1}{a_c+A}\left[\hat{x}-\left(\frac{a_I+A}{b}\right)(\hat{x}-(1-d)\hat{x})\right]=\frac{\hat{x}[b-d(a_I+A)]}{b(a_C+A)}$$

while $u(x,x')=0$. Thus, $u(\hat{x},\hat{x})-u(x,x')=\frac{\hat{x}[b-d(a_I+A)]}{b(a_C+A)}$, and we have

$$\frac{\hat{x}[b-d(a_I+A)]}{b(a_C+A)}\geqslant\hat{p}\,[b-d(a_I+A)]$$
$$\frac{\hat{x}}{b(a_C+A)}\geqslant\hat{p}.$$

Similarly, substituting the zero investment plan $M=(a_C+A,(1-d)(a_C+A))$, and

$$u(\hat{x},\hat{x})=1-\frac{1}{b}[\hat{x}-(1-d)\hat{x}]=1-\frac{d\hat{x}}{b}$$

while $u(x,x')=0$. Thus, $u(\hat{x},\hat{x})-u(x,x')=1-\frac{d\hat{x}}{b}$, and we have

$$1-\frac{d\hat{x}}{b}\geqslant 1+\hat{p}\,[(1-d)(a_C+A)-(a_C+A)]$$
$$\frac{d\hat{x}}{b}\leq\hat{p}\,[da_C+dA]$$
$$\frac{\hat{x}}{b(a_C+A)}\leq\hat{p}$$

Since $\frac{\hat{x}}{b(a_C+A)}\geqslant\hat{p}$ and $\frac{\hat{x}}{b(a_C+A)}\leq\hat{p}$, the golden rule price level is given by $\hat{p}=\frac{\hat{x}}{b(a_C+A)}$.

Proof of Proposition 2. We can rewrite ξ using the golden rule price system as $\xi=\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}$ since

$$\begin{aligned}\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}&=\frac{\frac{\hat{x}}{b(a_C+A)}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\frac{\hat{x}}{b(a_C+A)}}\\&=\frac{\hat{x}-(1-d)(a_C+A)}{(a_C+A)-\hat{x}}\\&=\frac{\left[\frac{b(a_C+A)}{b+d(a_C-a_I)}\right]-(1-d)(a_C+A)}{(a_C+A)-\left[\frac{b(a_C+A)}{b+d(a_C-a_I)}\right]}\\&=\frac{b(a_C+A)-(1-d)(a_C+A)[b+d(a_C-a_I)]}{(a_C+A)[b+d(a_C-a_I)]-b(a_C+A)}\\&=\frac{db(a_C+A)-(1-d)(a_C+A)d(a_C-a_I)}{(a_C+A)d(a_C-a_I)}\\&=\frac{b}{a_C-a_I}-(1-d)=\xi.\end{aligned}$$

Therefore, we obtain the equation of the line VM with some constant D as $x'=-\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)x+D$. Since the line VM goes through the golden rule stock $G=(\hat{x},\hat{x})$, we also have $\hat{x}=-\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)\hat{x}+D$. Substituting one into the other yields

$$\hat{x}=-\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)\hat{x}+x'+\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)x$$

$$\hat{x}\left[1+\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right]=x'+\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)x$$

$$\hat{x}\left[\frac{\left(\frac{d}{b}\right)}{\frac{1}{b}-\hat{p}}\right]=x'+\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)x$$

$$\hat{x}=\left[\frac{\frac{1}{b}-\hat{p}}{\frac{d}{b}}\right]x'+\left[\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{d}{b}}\right]x.$$

We now compute the value-loss of the VM line

$$\begin{aligned}\delta_{(\hat{p},\hat{x})}(x,x')&=u(\hat{x},\hat{x})-u(x,x')-\hat{p}\,(x'-x)\\&=\left[\frac{1-d\hat{x}}{b}\right]-\left[1-\frac{x'-(1-d)x}{b}\right]-\hat{p}x'+\hat{p}x\\&=\left[1-\left(\frac{1}{b}-\hat{p}\right)x'-\left(\hat{p}-\left(\frac{1-d}{b}\right)x\right)\right]-\left[1-\frac{x'-(1-d)x}{b}\right]\\&\quad-\hat{p}x'+\hat{p}x=0.\end{aligned}$$

Proof of Proposition 3. We also show that the value-loss of a plan on $V'M'$ can be measured by the difference of the golden rule utility level and the utility level of a plan at which $V'M'$ and the 45° line intersects, such as $T=(x^T,x^T)$ as in Fig. 3. The equation of $V'M'$ is obtained with some constant D such that $x'=-\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)x+D$. With the same procedure as before, we have $x^T=-\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)x^T+D$. Substituting one into the other yields

$$\begin{aligned}x^T&=-\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)x^T+x'+\left(\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{1}{b}-\hat{p}}\right)x\\&=\left[\frac{\frac{1}{b}-\hat{p}}{\frac{d}{b}}\right]x'+\left[\frac{\hat{p}-\left(\frac{1-d}{b}\right)}{\frac{d}{b}}\right]x.\end{aligned}$$

Therefore,

$$\begin{aligned}u\left(x^T,x^T\right)&=1-\frac{x^T-(1-d)x^T}{b}=1-\frac{d}{b}x^T\\&=1-\left(\frac{1}{b}-\hat{p}\right)x'-\left[\hat{p}-\left(\frac{1-d}{b}\right)\right]x.\end{aligned}$$

For any plan on $V'M'$, the value-loss is thus given by

$$\begin{aligned}\delta_{(\hat{p},\hat{x})}(x,x') &= u(\hat{x},\hat{x})-\left[1-\frac{x'-(1-d)x}{b}\right]-\hat{p}\,(x'-x)\\ &= u(\hat{x},\hat{x})-1+\frac{x'}{b}-\frac{(1-d)x}{b}-\hat{p}x'+\hat{p}x\\ &= u(\hat{x},\hat{x})-\left[1-\left(\frac{1}{b}-\hat{p}\right)x'-\left[\hat{p}-\left(\frac{1-d}{b}\right)\right]x\right]\\ &= u(\hat{x},\hat{x})-u\left(x^T,x^T\right).\end{aligned}$$

It is clear that any plan on $V'M'$ has the same value-loss. Since the utility level of a plan such as T is always less than the golden rule utility level, the value-loss is always positive. If $V'M'$ shifts outwards from VM, then a point T gets far away from G which implies that the value-loss increases. The same argument goes through for any plan in the triangle VPM.

Proof of Proposition 4. In the area VPM, the value-loss of a plan is given by

$$\delta_{(\hat{p},\hat{x})}(x,x') = u(\hat{x},\hat{x})-\frac{1}{(a_c+A)}\left[x-\left(\frac{a_I+A}{b}\right)(x'-(1-d)x)\right]-\hat{p}\,(x'-x).$$

Then a marginal change in the value-loss with respect to x', for any given x, is

$$\frac{\partial\delta_{(\hat{p},\hat{x})}(x,x')}{\partial x'} = \frac{a_I+A}{b(a_c+A)}-\hat{p} = \frac{a_I+A-\hat{x}}{b(a_c+A)}.$$

We can see that in the case of the labor-intensive consumption good, $\frac{a_I+A-\hat{x}}{b(a_c+A)}>0$. Therefore, if a plan deviates vertically downwards from the plan on VM the value-loss decreases, and if it deviates by $\Delta x'>0$ the value-loss associated with that plan is $\Delta x'\left[\frac{a_I+A}{b(a_c+A)}-\hat{p}\right]>0$. On the other hand, in the region $LMVN$, the value-loss of any plan in this area is given by

$$\delta_{(\hat{p},\hat{x})}(x,x') = u(\hat{x},\hat{x})-\left[1-\frac{x'-(1-d)x}{b}\right]-\hat{p}\,(x'-x).$$

Then a marginal change in the value-loss with respect to x', for any given x, is given by

$$\frac{\partial\delta_{(\hat{p},\hat{x})}(x,x')}{\partial x'} = \frac{1}{b}-\hat{p} = \frac{1}{b}-\frac{\hat{x}}{b(a_c+A)}.$$

We can see that in the case of the labor-intensive consumption good $\frac{1}{b}-\hat{p}<0$. Therefore, if a plan deviates vertically upwards from the plan on VM, the value-loss decreases. If a plan deviates by $\Delta x'<0$ the value-loss associated with that plan is $\Delta x'\left(\frac{1}{b}-\hat{p}\right)>0$.

Proof of Proposition 5. Pick any two plans, say P_1 and P_2 on a horizontal line beginning at G as shown in Fig. 4. The sum of these two plans is indicated by a plan P_3. Since ΔP_1GM_1 and $\Delta P'_3M_2M_3$ are congruent, the length of P_3M_3 is the sum of P_1M_1 and P_2M_2. This length stands for the magnitude of deviation from the zero-value-loss plan, which is $\Delta x'$. Then, a value-loss of the sum of two plans is the sum of the value-losses of two plans.

Proof of Proposition 6. Consider Fig. 5. There exists a small ε, such that if the initial stock $x=\hat{x}+\varepsilon$, the final stock is $x'=-\xi(\hat{x}+\varepsilon)+\frac{b(a_C+A)}{ac-a_I}$. If $x'>x$, then we have

$$\begin{aligned}-\xi(\hat{x}+\varepsilon)+\frac{b(a_C+A)}{ac-a_I}&>\hat{x}+\varepsilon\\ \frac{b(a_C+A)}{(ac-a_I)}&>(\hat{x}+\varepsilon)(1+\xi).\end{aligned}$$

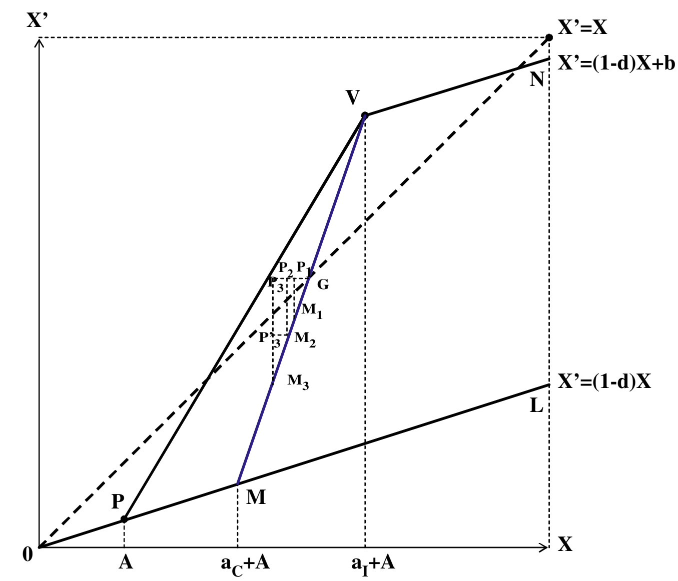

Fig. 4. The sum of the value-losses when $a_I>a_C$.

Substituting $\hat{x}=\frac{b(a_C+A)}{b+d(a_C-a_I)}$, and $\xi=\frac{b}{a_C-a_I}-(1-d)$, we have

$$\begin{aligned}\frac{b(a_C+A)}{(ac-a_I)}&>\left[\frac{b(a_C+A)}{b+d(a_C-a_I)}+\varepsilon\right]\left(1+\frac{b-(1-d)(a_C-a_I)}{a_C-a_I}\right)\\ \frac{b(a_C+A)}{(ac-a_I)}&>\left[\frac{b(a_C+A)+\varepsilon[b+d(a_C-a_I)]}{b+d(a_C-a_I)}\right]\left[\frac{b+d(a_C-a_I)}{a_C-a_I}\right]\\ b(a_C+A)&>b(a_C+A)+\varepsilon[b+d(a_C-a_I)]\\ 0&>\varepsilon[b+d(a_C-a_I)].\end{aligned}$$

Which is true if $b<d(a_C-a_I)$.

On the other hand, if the initial stock is $x=\hat{x}-\varepsilon$, the final stock is $x'=-\xi(\hat{x}-\varepsilon)+\frac{b(a_C+A)}{ac-a_I}$. If $x'<x$, then we have

$$\begin{aligned}-\xi(\hat{x}-\varepsilon)+\frac{b(a_C+A)}{ac-a_I}&<\hat{x}-\varepsilon\\ \frac{b(a_C+A)d}{(ac-a_I)d}&<(\hat{x}-\varepsilon)(1+\xi).\end{aligned}$$

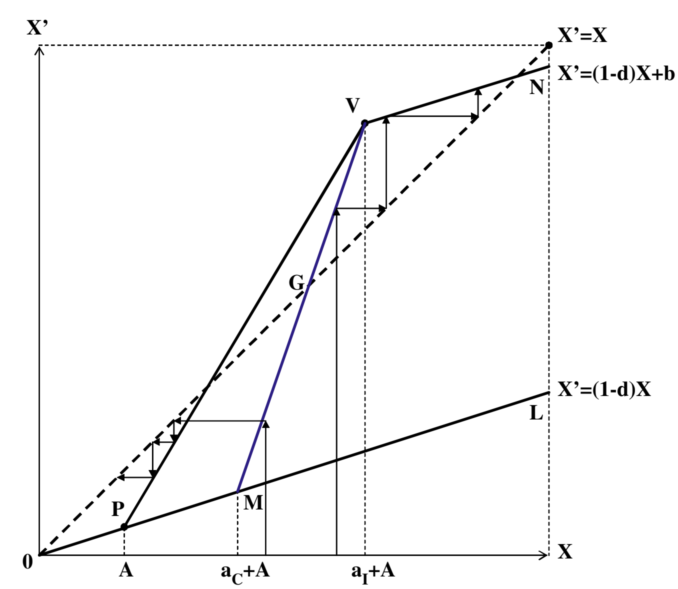

Fig. 5. $\xi<-1$.

Substituting $\hat{x}$ and ξ, then we have

$$\frac{b(a_C+A)d}{(ac-a_I)d} < \left[\frac{b(a_C+A)}{b+d(a_C-a_I)}-\varepsilon\right]\left[1+\frac{b-(1-d)(a_C-a_I)}{a_C-a_I}\right]$$
$$\frac{b(a_C+A)d}{(ac-a_I)d} < \left[\frac{b(a_C+A)-\varepsilon[b+d(a_C-a_I)]}{b+d(a_C-a_I)}\right]\left[\frac{b+d(a_C-a_I)}{a_C-a_I}\right]$$
$$b(a_C+A) < b(a_C+A)-\varepsilon[b+d(a_C-a_I)]$$
$$0 < -\varepsilon[b+d(a_C-a_I)].$$

Which is true if $b<d(a_C-a_I)$.

Therefore, whenever you start from an initial stock slightly above or below the golden rule stock, the final capital stock gets further away from the golden rule stock. Only, when you start from the golden rule stock, then you remain in the golden rule stock. Accordingly, the McKenzie facet shrinks to the von-Neumann point.

Proof of Proposition 7. In Fig. 6, consider the case where $x(0)\in\left[\hat{x},\frac{\hat{x}}{1-d}\right]$. Pick any initial stock, say x_1. It is not optimal to take a plan strictly above the 45° line, since such a plan lets the level of capital stock in the next period gets farther away from the golden rule stock. Consider a plan below the GQ' line. Since the value-loss associated with such a plan is larger than the one with a plan on GQ', it is always better to take a plan on GQ' and converge to the golden rule stock immediately. Thus, the optimal plan lies between the 45° line and GQ' line for x_1. If we take a plan S_1 on GQ', we suffer value-loss only in the first period and then stay on the golden rule stock afterwards with zero value-loss. If we take a plan strictly above GQ', then the value-loss is less than the one associated with S_1, but we still have to have some value-loss to get to the golden rule stock in the future. Thus, it takes more than two periods to converge to the golden rule stock.

Consider the following two programs: one is to take a plan S_1 on GQ', and the other is to take a plan above GQ', say S_2, in the first period and then take a plan S_3 on GQ' in the next. For simplicity, let the distance between $\hat{x}$ and x_1 be unity. Then the aggregate value-loss of the former program is given by

$$\delta(x_1,\hat{x}) = \left[1-\frac{\hat{x}-(1-d)x_1}{b}\right]-\left[1-\frac{\hat{x}-(1-d)\hat{x}}{b}\right]-\hat{p}\,(\hat{x}-x_1)$$
$$= \hat{p}-\frac{1-d}{b} = \xi\left[\frac{1}{b}-\hat{p}\right] = \xi\left[\frac{1}{b}-\frac{\hat{x}}{b(a_c+A)}\right] = \xi\left[\frac{(a_c+A)-\hat{x}}{b(a_c+A)}\right].$$

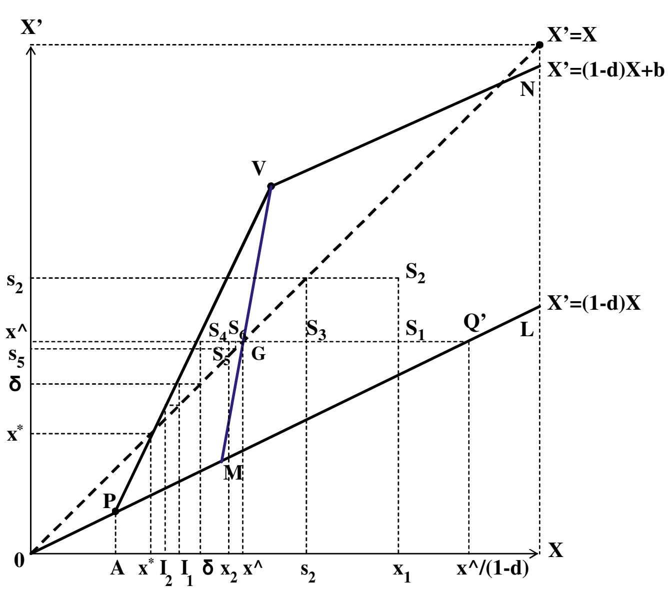

Fig. 6. The optimal policy if $\xi<-1$.

The aggregate value-loss of the latter is given by $\delta(x_1,s_2)+\delta(s_2,\hat{x})$. The first term is given by

$$\delta(x_1,s_2) = \left[1-\frac{s_2-(1-d)x_1}{b}\right]-\left[1-\frac{\hat{x}-(1-d)\hat{x}}{b}\right]-\hat{p}\,(s_2-x_1)$$
$$= \frac{-s_2+(1-d)x_1+d\hat{x}}{b}-\hat{p}\,(s_2-x_1).$$

The second term is given by

$$\delta(s_2,\hat{x}) = \left[1-\frac{\hat{x}-(1-d)s_2}{b}\right]-\left[1-\frac{\hat{x}-(1-d)\hat{x}}{b}\right]-\hat{p}\,(\hat{x}-s_2)$$
$$= \frac{-\hat{x}+(1-d)s_2+d\hat{x}}{b}-\hat{p}\,(\hat{x}-s_2).$$

Thus, the sum is given by

$$\delta(x_1,s_2)+\delta(s_2,\hat{x}) = \frac{(1-d)+d(\hat{x}-s_2)}{b}+\hat{p}$$
$$= \left[\hat{p}-\frac{(1-d)}{b}\right]+\frac{d}{b}(s_2-\hat{x})\frac{\frac{1}{b}-\hat{p}}{\frac{1}{b}-\hat{p}}$$
$$= \left[\hat{p}-\frac{(1-d)}{b}\right]+(s_2-\hat{x})\left(\frac{1}{b}-\hat{p}\right)\left(1+\frac{\hat{p}-\frac{1-d}{b}}{\frac{1}{b}-\hat{p}}\right)$$
$$= \left[\hat{p}-\frac{(1-d)}{b}\right]+(s_2-\hat{x})\left(\frac{1}{b}-\hat{p}\right)(1+\xi)$$
$$= \xi\left[\frac{(a_c+A)-\hat{x}}{b(a_c+A)}\right]+(s_2-\hat{x})\left[\frac{(a_c+A)-\hat{x}}{b(a_c+A)}\right](1+\xi).$$

Since $\frac{(s_2-\hat{x})(\xi+1)(a_C+A-\hat{x})}{b(a_C+A)}>0$ because $\xi<-1$ and $a_C+A<\hat{x}$, the aggregate value-loss associated with the former is less than that with the latter.

Next, consider a program starting from x_1 and taking more than two periods, say n, to get to the golden rule stock. Then, we can apply the same argument to the period $n-1$ and the aggregate value-loss decreases if we get to the golden rule stock in $n-1$ periods. Hence, we can conclude that the aggregate value-loss associated with a program converging to the golden rule stock in the first period is less than any other program which takes more than two but a finite number of periods to get to the golden rule stock. Next, consider a possibility that a program converges to the golden rule stock in an infinite number of periods. We can construct such a program with an arbitrary sequence $e_1, e_2, \ldots$ where $e_i>0$ for every i as follows; in the first period it takes a plan of which x'-ordinate is greater than $\hat{x}$ by $e_1>0$, in the second period by $e_2>0$ and this is repeated infinitely. Since it is never optimal to take a plan over the 45° line, $e_1>e_2>\ldots$. The aggregate value-loss is $(\xi+e_1+e_1\xi+e_2+e_2\xi+e_3+\ldots)\,\frac{(a_C+A-\hat{x})}{b(a_C+A)} = \left(\xi+(\xi+1)\sum_{i=1}^{\infty}e_i\right)\frac{(a_C+A-\hat{x})}{b(a_C+A)}$. Even if $\sum_{i=1}^{\infty}e_i$ diverges or converges to some finite positive value, the aggregate value-loss is larger than $\frac{\xi(a_C+A-\hat{x})}{b(a_C+A)}$ since $\xi<-1$. Thus it is optimal for any initial stock $x(0)\in\left[\hat{x},\frac{\hat{x}}{1-d}\right]$ to converge to the golden rule stock immediately.

Moving to the region $x\in\left[\delta=\frac{\hat{x}+\frac{b}{a_I}A}{\left(\frac{b}{a_I}+1-d\right)},\hat{x}\right]$. With the same argument as above the optimal plan must lie between 45° line and GQ' line. Pick any initial capital stock, say x_2, and consider the following two programs: one is to take a plan S_4 on GQ',and the other is to take S_5 below GQ' in the first period and then a plan S_6 on GQ'

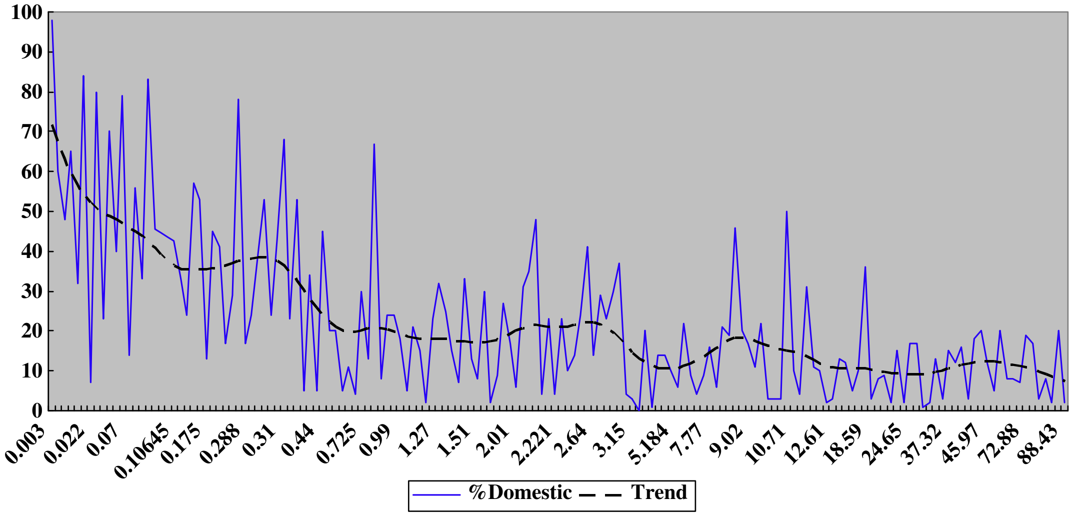

Fig. 7. The percentage used for domestic purposes out of total water withdrawals in cubic kilometers.

in the next period. Let the distance between $\hat{x}$ and x_2 be unity. The aggregate value-loss of the former program is $\frac{-\xi(a_I + A - \hat{x})}{b(a_C + A)}$, and that of the latter is $\frac{(-\xi - (\hat{x} - s_5) - \xi(\hat{x} - s_5))(a_I + A - \hat{x})}{b(a_C + A)}$. Since $\frac{(\hat{x} - s_5)(\xi + 1)(a_I + A - \hat{x})}{b(a_C + A)} < 0$ because $\xi < -1$ and $a_I + A > \hat{x}$, the aggregate value-loss associated with the former program is less than that with the latter. Applying the same argument above, we can conclude that it is optimal for any initial stock $x(0) \in \left[\frac{\hat{x} + \frac{b}{a_I}A}{\left(\frac{b}{a_I} + 1 - d\right)}, \hat{x}\right]$ to converge to the golden rule stock immediately.

All is left is to characterize the optimal policy for $x \in \left(x^*, \frac{\hat{x} + \frac{b}{a_I}A}{\left(\frac{b}{a_I} + 1 - d\right)}\right)$ and $x \in \left[\frac{\hat{x}}{(1-d)}, \infty\right)$. We can solve for x^* as the intersection between the *PV* line and the 45° line. Since the equation of the *PV* line is given by

$$x' = \left[(1-d) + \frac{b}{a_I}\right]x - \frac{b}{a_I}A.$$

Then, $x^* = \frac{\frac{b}{a_I}A}{\left(\frac{b}{a_I} - d\right)}$. We only consider the case $x \in \left(x^*, \frac{\hat{x} + \frac{b}{a_I}A}{\left(\frac{b}{a_I} + 1 - d\right)}\right)$. Let's read off $I_1, I_2, \ldots$to the left of $\frac{\hat{x} + \frac{b}{a_I}A}{\left(\frac{b}{a_I} + 1 - d\right)}$.[2] For $x \in \left(I_1, \frac{\hat{x} + \frac{b}{a_I}A}{\left(\frac{b}{a_I} + 1 - d\right)}\right)$, if the program takes a plan of which x'-ordinate is greater than or equal to $\frac{\hat{x} + \frac{b}{a_I}A}{\left(\frac{b}{a_I} + 1 - d\right)}$, it takes two periods to get to the golden rule stock. If a program takes a plan of which x'-ordinate is below it, it takes more than three periods and it is easy to see that the aggregate value-loss of such a program is larger than that of any program which takes two periods to get to the golden rule stock. Let's consider the following two possible programs starting from any but the same initial stock in $\left(I_1, \frac{\hat{x} + \frac{b}{a_I}A}{\left(\frac{b}{a_I} + 1 - d\right)}\right)$. The first program is to take a plan on *PV*, and the second is to take any plan below *PV*. Let a distance between the x'-ordinate of the two plans in the first period be unity. The difference between these two programs is that the former suffers higher value-loss in the first period than the latter by $\frac{(a_I + A - \hat{x})}{b(a_C + A)}$ and lower in the second by $\frac{-\xi(a_I + A - \hat{x})}{b(a_C + A)}$. Thus the former program has a lower value-loss since $\xi < -1$. The same argument goes through for any nth interval $[I_n, I_{n-1})$. For $x \in \left[\frac{\hat{x}}{(1-d)}, \infty\right]$, the same method can be applied and it is optimal to take a plan on *PL*.

Proof of Proposition 8. The initial capital stock x^* occurs where the *PV* intersects with the 45° line. Therefore, for any $x < x^*$, we have $x' < x^*$, and the capital stock converges to zero.

Data

The dataset used is extracted from the World Resources Institute. Data were collected over a period of 15 to 25 years and compiled in 2003 by the FAO AQUASTAT global information system of water and agriculture. The current AQUASTAT database provides data per 5-year period if available. Data presented by World Resources Institute is the most recent value which is the period from 1998 to 2002 in the AQUASTAT database. The first variable extracted is the annual total water withdrawals which is the gross amount of water extracted from any source, either permanently or temporarily, for a given use. It can be either diverted towards distribution networks or directly used. It includes consumptive use, conveyance losses, and return flow. The second variable extracted is the percent of water withdrawals used for domestic purposes, which refers to the proportion of total water withdrawals that is allocated to domestic uses, which include drinking water plus water withdrawn for homes, municipalities, commercial establishments, and public services. The third variable extracted is the percent of water withdrawals used for agricultural purposes, which refers to the proportion of total water withdrawals that is allocated to the agricultural sector, primarily for irrigation. The fourth variable extracted is the percent of water withdrawals used for industrial purposes, which refers to the proportion of total water withdrawals that is allocated to industrial uses, which include cooling machinery

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[2] $$I_n = \frac{\hat{x} + \frac{b}{a_I}A}{\left(\frac{b}{a_I} + 1 - d\right)^{n+1}}.$$

and equipment, producing energy, cleaning and washing goods produced as ingredients in manufactured items, and as a solvent.

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