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UNDISCOUNTED OPTIMAL GROWTH WITH CONSUMABLE CAPITAL: APPLICATION TO WATER RESOURCES

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This paper utilizes the geometric techniques developed in Khan and Mitra (2005, 2007) to analyze the optimal intertemporal allocation of water resources in a dynamic setup without discounting. The framework features two sectors: the first uses labor to purify water, while the second uses labor and purified water for irrigation to produce an agricultural consumption good. Purified water can also be used as potable water for drinking purposes. The planner allocates the available factors of production between the two sectors every period, and determines the optimal amounts of purified water, potable water, and irrigation water. The geometry characterizes the optimal path depending on whether the irrigation sector is more labor intensive than the purification sector. When the irrigation sector is labor intensive, the optimal path is a non converging cycle around the golden rule stock of purified water, while if the purification sector is labor intensive, there is a damped cyclical convergence to the golden rule stock.

JEL classification codes: C61, D90, Q25
Key words: water resources, overtaking criterion, golden rule stock

I. Introduction

Ramsey (1928) addressed the problem of undiscounted optimal growth in which the optimal program of capital accumulation was derived from the maximization of a utility sum over an infinite time period. Samuelson and Solow (1956) extended

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the framework to a model with many capital goods. Atsumi (1965) and Von Weiszsaker (1965) proposed the overtaking criterion approach. Under this criterion, Gale (1967) and Brock (1970) showed the existence of an optimal path. Approaching a special case of the two sector model proposed by Nishimura and Yano (1995, 1996, 2000), Fujio (2005, 2008) employed geometric techniques developed in Khan and Mitra (2007) to characterize the optimal policy. Khalifa (2009) employs the same geometric techniques in an extension of Fujio (2005, 2008) to consider the case of consumable capital; that is both the consumption good and capital can be consumed. The framework extends the Leontief two sector optimal growth model, which analyzes the optimal allocation of capital and labor to a consumption sector and an investment sector in every period. However, the planner has preferences defined over both the consumption good and the consumable capital. The utility function with two arguments (the consumption good and the consumable capital) has implications on the planner's preferences, on the value-loss and on the optimal program.

This paper considers a specific application, to the case of water resources. Water can be used in irrigation to produce an agricultural consumption good, and can also be used directly for drinking purposes. Therefore, water can serve as capital that is used in the production process of a consumption good, and can be consumable as well. Worldwide, water used in agriculture and industry accounts for 90% while domestic use accounts for 10% of total water withdrawal (WRI, 2003). The percentage of water withdrawals used for domestic purposes in the developed world is close to that used in the developing world. However, the latter has a higher percentage of withdrawals used in agriculture, relative to industry, compared to the former. Accordingly, the model features two sectors. The first utilizes labor to purify water. Purified water can either be used for drinking purposes, or in an irrigation sector along with labor to produce an agricultural consumption good.

We utilize the techniques developed by McKenzie (1968) and Brock (1970), or alternatively the reduced form without discounting. Addressing the optimal intertemporal allocation of water resources without discounting guarantees intergenerational equity. In that case, we assume that future welfare levels are treated equally as current ones. The discount factor takes the value of one. This is motivated by Khan (2002) who noted that "to ensure sustainability and intergenerational equity in managing natural resources, the optimal intertemporal allocation of resources available to any collective ought not to be based on criteria that discount the weight that is attached to future generations or cohorts of that collective." This is also motivated by the fact that the exploding world population is expected to continue to exert an enormous pressure on the scarce water resources. The available resources have not only been depleted and over exploited in an attempt to satisfy the increasing demand, but have been unequally distributed and inefficiently utilized as well. Moreover, water resources are not always contained within the boundaries of the countries dependent on them, but are usually shared by a number of nations. Thus, the upsurge in demand for water transformed the problem into a survival issue and a possible catalyst for future conflicts as countries dependent on external water resources reorient their national strategies towards the protection of their rights to water access by all means possible. In this context, Yoffe et al. (2004) use the Transboundary Freshwater Dispute Database to provide evidence that the "likelihood of intense dispute rises as the average precipitation within a basin decreases."

Finally, this paper applies the techniques of Euclidean geometry developed by Khan and Mitra (2005, 2007) to provide insights into the optimal paths and characterize the optimal policy. The geometric analysis characterizes the optimal path depending on whether the irrigation sector is more labor intensive than the purified sector. In the former case, the optimal path is a nonconverging cycle around the golden rule stock of purified water, while in the latter case it exhibits a damped cyclical convergence to the golden rule stock. In this context, each trajectory is considered optimal in its specific case. Therefore, it is optimal to follow a nonconverging cycle when the irrigation sector is labor intensive, while it is optimal to follow a converging cycle if the water purification sector is labor intensive. This implies that the characterization of the optimal program has policy implications, as the model specifies the optimal program to be adopted to ensure sustainability of the scarce water resources.

The remainder of the paper is organized as follows: Section II presents the model and Section III contains our conclusions.

II. Model

A. Technology

Our economy is represented by two sectors: water purification and irrigation. For each date $t \in \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers, production of one unit of purified water requires a units of labor. One unit of labor and one unit of purified water used in the irrigation sector produce one unit of the consumption good. Moreover, purified water is consumed for drinking purposes. In every period, a planner allocates a given amount of purified water and labor to either sector. There is one technology available to each sector. For each date t, let the technologies be stationary and given by

$$c(t+1)=\min\{w_r(t),l_r(t)\}, \quad (1)$$

$$z(t+1)=\frac{1}{a}\cdot l_p(t), \quad (2)$$

where $c(t+1)$ and $z(t+1)$ denote the amounts of the consumption good and the investment in water purification in period $t+1$, respectively. $l_r(t)$ and $l_p(t)$ denote the labor employed in the irrigation sector for the production of the consumption good and the labor employed in the water purification sector, respectively. $w_r(t)$ denotes the amount of purified water used in the irrigation sector. The amount of purified water used for drinking purposes in period t, or what is referred to as potable water, is denoted $w_p(t)$.

We also assume that the stock of purified water evaporates at a rate $e\in(0,1)$. In other words, from a given stock of purified water, a portion is used in both sectors and the leftover evaporates at the given rate. The residual stock plus the purified water produced in the same period form next period's purified water stock. If $x(t) \geq 0$ denotes purified water stock available in period t, then

$$x(t+1)=(1-e)\cdot[x(t)-w_p(t)-w_r(t)]+z(t+1). \quad (3)$$

Therefore, investment in water purification replaces the evaporated water and the amounts used for drinking and irrigation in the previous period.

Labor is normalized to unity in every period of time. The gross increase in purified water stock $z(t+1)$ requires $l_p(t)=a\cdot z(t+1)$ units of labor in period t. The labor required in the irrigation sector $l_r(t)= w_r(t)$. Therefore, the labor constraint is given by

$$0\leq a\cdot z(t+1)+w_r(t)=l_p(t)+l_r(t)\leq 1. \quad (4)$$

Definition 1: *A program from $x_0\in\Re_+$ is a sequence $\{x(t),w_r(t),w_p(t)\}$ with $(x(t),w_r(t),w_p(t))\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+$ such that $x(0) = x_0$, and for all $t\in\mathbb{N}$,:*

$$x(t+1)\geq(1-e)[x(t)-w_p(t)-w_r(t)],\ 0\leq w_p(t)+w_r(t)\leq x(t),$$

$$0\leq a[x(t+1)-(1-e)[x(t)-w_p(t)-w_r(t)]]+w_r(t)\leq 1.$$

Definition 2: *Associated with any program* $\{x(t), w_r(t), w_p(t)\}$ *is a gross stock increase sequence* $\{z(t+1)\}$ *with* $z(t+1) \in \mathbb{R}_+$, *and a consumption sequence* $\{c(t+1)\}$ *with* $c(t+1) \in \mathbb{R}_+$, *such that for all* $t \in \mathbb{N}$,:

$$z(t+1) = x(t+1) - (1-e) \cdot \left[x(t) - w_r(t) - w_p(t)\right],\ c(t+1) = \min\left\{w_r(t), l_r(t)\right\}.$$

Definition 3: *A program* $\{x(t), w_r(t), w_p(t)\}$ *is called stationary if for all* $t \in \mathbb{N}$,:

$$\left\{x(t), w_r(t), w_p(t)\right\} = \left\{x(t+1), w_r(t+1), w_p(t+1)\right\}.$$

B. Preferences

The preferences of the planner are represented by a felicity function $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. We assume that the felicity function $v(c(\mathrm{t}), w_p(\mathrm{t})) = v(w_r(\mathrm{t}), w_p(\mathrm{t}))$ is separable in its two arguments: potable water and the consumption good. Future welfare levels are treated equally as current ones; that is the discount factor is assumed unity. Following Khan and Mitra (2005), we work with the overtaking criterion of optimality.

Definition 4: *A program* $\left\{x^*(t), w_r^*(t), w_p^*(t)\right\}_{t=0}^{\infty}$ *starting from* $x(0)$ *is called optimal if there does not exist any* $\varepsilon > 0$ *and a time period* t_ε *such that* $\sum_{t=1}^{T}\left[v(w_r(t), w_p(t)) - v(w_r^*(t), w_p^*(t))\right] \geq \varepsilon$ *for all* $T \geq t_\varepsilon$.

That is an optimal program is one in comparison to which no other program for the same initial stock is eventually significantly better, for any given level of significance. A program is said to be stationary if it is constant over time. A program is said to be a stationary optimal program if it is stationary and optimal.

C. Reduced form

Following McKenzie(1968), we convert the above model into its reduced form. The latter is summarized by the transition possibility set Ω as a collection of pairs (x, x'), such that it is possible to have x' of the amounts of the purified water in the next period from the x amounts of purified water available in the current period. Formally,

$$\Omega = \left\{(x, x') \in \mathbb{R}_+ \times \mathbb{R}_+ : x' - (1-e)\cdot(x - w_r - w_p) \geq 0 \right.$$
$$\left. \text{and } a(x' - (1-e)\cdot(x - w_r - w_p)) \leq 1\right\}. \qquad (5)$$

Using this formulation, we can keep track of the transition dynamics of only the state variable, x(t), overtime. For any $(x,x')\in\Omega$, one can consider the amounts (w_p,w_r) available for drinking and irrigation purposes, respectively.

Formally, we have a correspondence $\Psi:\Omega\rightarrow\mathbb{R}_+\times\mathbb{R}_+$, given by

$$\Psi(x,x')=\{(w_p;w_r)\in\Re_+\times\Re_+:0\leq w_p+w_r\leq x \text{ and } w_r\leq 1-a(x'-(1-e)\cdot(x-w_r-w_p))\}. \quad (6)$$

Finally, the reduced form utility function, $u:\Omega\rightarrow\mathbb{R}_+$, is defined on Ω such that

$$u(x,x')=\max\{v(w_p,w_r):(w_p,w_r)\in\Psi(x,x')\}. \quad (7)$$

D. Geometry

In Figure 1A, the x-axis stands for the amount of purified water available today, while the x'-axis stands for the amount of purified water available tomorrow. To proceed with the geometric analysis, we impose the following assumptions on the felicity function with respect to its two arguments (potable water and the consumption good): (1) the marginal utility of the consumption good is constant, and as the amount of the consumption good is equal to the amount of irrigation water, then the marginal utility of purified water used in irrigation is constant as well; (2) the marginal utility of purified water used for drinking is larger than the marginal utility of water used in irrigation up to a threshold $x = A < (1/a)$; (3) the marginal utility of potable water is negative after the threshold $w_p = A$.

These assumptions ensure that if the initial stock is less than the threshold A, the planner prefers to use all the available stock for drinking, and none for irrigation. While if the initial stock is higher than the specified threshold, the planner prefers to drink an amount that exactly equals the threshold A only. These assumptions can be justified by the actual percentages of withdrawals dedicated to domestic purposes corresponding to every level of total water withdrawals, in addition to the trend. The declining trend can provide a justification to our assumption that if the available stock of water is lower than a threshold A, the planner prefers to use all the available stock for drinking and accordingly the percentage that is used for water consumption is high. However, as the available stock gets larger than the threshold, the planner prefers to use the exact amount A for drinking purposes. This means that as the stock gets larger, the percentage dedicated for water consumption gets smaller, consistently with the observed trend. This ensures that the main assumptions of the model are based upon real world observations.

In this context, the line *OVM* plays a central role in our geometric investigation. First of all, it is referred to as the full employment-no excess capacity line, as it provides us with the set of plans with fully utilized labor and purified water. At the segment with an initial stock of purified water $x \in (0, A]$, the planner prefers to use all the available purified water today for the purpose of drinking. This is because the utility of using an additional unit of purified water for drinking is higher than the utility of using it for irrigation, as long as the initial amount available is less than A. Then, the planner uses all the available labor in purifying $(1/a)$ units of water for the next period. That is why the line *OV* is horizontal at $x' = (1/a)$ for all $x \in (0, A]$. If the initial stock of purified water $x(0) \in (A, 1+A]$, the planner decides to use only a portion A from the available purified water today for drinking purposes. The leftover $(x\text{-}A)$ is combined with labor in the irrigation sector to produce the consumption good. The remaining labor $[1\text{-}(x\text{-}A)]$ are employed to invest $z = (1/a) \cdot [1-(x-A)] = x'$ of purified water for the next period. At $x = 1+A$, the planner consumes A and uses the remaining 1 unit of water with all the available labor (that is normalized to unity) in the irrigation sector, in order to produce the consumption good. Nothing is left for investment, and thus the planner ends up with $z = 0 = x'$. It is clear in Figure 1A that the slope of the full employment-no excess capacity line is zero at *OV*, while the slope of the *VM* line is given by

$$-\frac{VQ}{QM} = -\frac{\left(1/a\right)}{(1+A)-A} = -\frac{1}{a} \equiv \xi. \tag{8}$$

To obtain the equation of the line *VM*, we pick any point, say $S = (x, x')$ as in Figure 1A. Since ΔMVQ and ΔMSS_x are similar triangles, we have

$$\frac{x'}{\left(1/a\right)} = \frac{SS_x}{VQ} = \frac{MS_x}{MQ} = \frac{(1+A)-x}{(1+A)-A} = 1+A-x. \tag{9}$$

Thus, the equation for the *VM* line is given by

$$x' = \frac{1}{a}(1+A-x) = \left(\frac{1+A}{a}\right) - \left(\frac{1}{a}\right) \cdot x = \left(\frac{1+A}{a}\right) + \xi . x \ . \tag{10}$$

The equation for the full employment-no excess capacity line *OVM* is

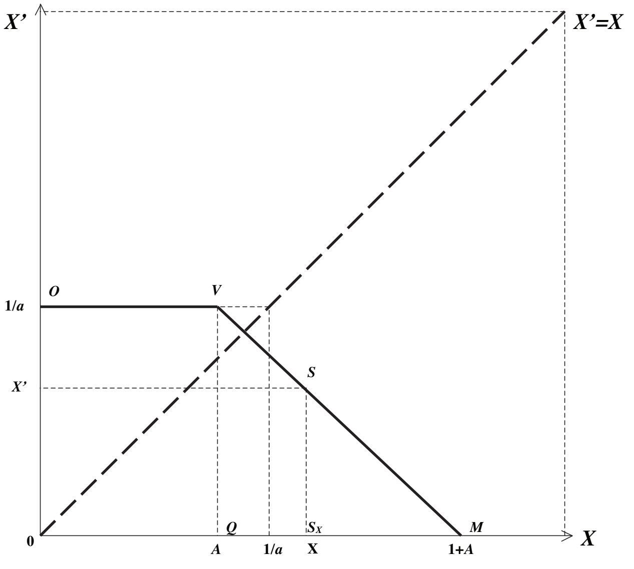
$$x' = \begin{cases} \frac{1}{a} \ \ for \ \ all \ \ x \in (0, A] \\ \left(\frac{1+A}{a}\right) + \xi x \ \ for \ \ all \ \ x \in (A, 1+A]. \end{cases} \tag{11}$$

If $x \in (0, A]$, the consumption of potable water is at a level of $w_p = x$, with a zero level of the consumption good. Therefore, the consumption good is not influencing the utility of the planner in this segment which is solely determined by the amount of potable water. On the other hand, if $x(0) \in (A, 1 + A]$, the consumption of potable water is constant at A. Therefore, potable water is not influencing the utility of the planner in this segment which is solely determined by the amount of the consumption good. Therefore, the planner's indifference map over (x,x')-plane can be easily configured.

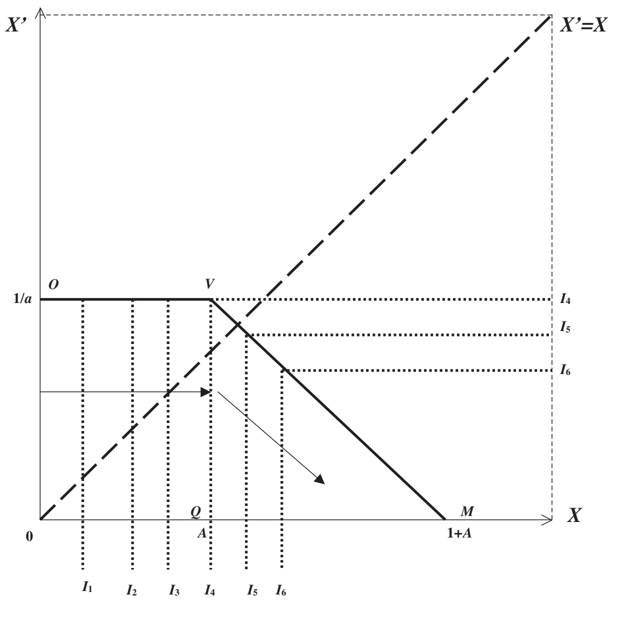
As we can see in Figure 1B, the indifference map of the planner if $x \in (0, A]$ is a set of vertical curves parallel to I_1. The reason is that at this segment, the level of potable water in any plan in an indifference curve is exactly equal to the available purified water x while the level of the consumption good equals to zero. In addition, any horizontal movement to the right of a plan on an indifference curve is accompanied by a higher level of potable water and accordingly a higher level of utility. The indifference curves are, however, kinked on the *VM* line as shown in Figure 1B. Consider the line I_4 on the *VM* line. By specification and construction, the amounts of labor and purified water used in the consumption good sector are all identical to the amount of the consumption good. If we move vertically downwards on I_4 from the highest x', this implies a decrease in the terminal stock of purified water x' and a consequent increase in the labor available to the consumption good sector. However, this sector is constrained by available purified water and hence no increase in the consumption good and thereby in utility is possible. On the other hand, consider a horizontal move from the highest x' on I_4 in a direction parallel to *QM*. This move represents an increase in the initial purified water stock x and a consequent increase in the purified water available to the consumption good sector. But no labor is being released and this sector now faces a labor constraint. Hence, no increase in either the consumption good or in utility is possible. Obviously, because the planner chooses to drink an exact amount A of potable water, it does not affect our indifference curves. Thus, we obtain an indifference curve I_4 of the Leontief type with a kink at its intersection with *VM*. If the initial purified water stock is increased and the terminal stock decreased, relative to I_4, as at I_5 and I_6 for instance, an increased amount of both purified water and labor is made available to the consumption sector, resulting in a higher level of the consumption good and thereby utility. The line *VM* thus takes on a new identity; it pegs a map of kinked indifference curves in the initial-terminal purified water (x,x')-plane.

Figure 1. The model geometry

A. Full employment-no excess capacity line



B. Planner's preference map



E. Benchmarks

We first derive a golden rule capital stock, $\hat{x}$, and a golden rule price level, $\hat{p}$, and then we define a value loss for every single plan $(x,x')\in\Omega$ in terms of $\hat{x}$ and $\hat{p}$. Second, we show that the *VM* line is the zero-value-loss line and that lines parallel to *VM* are iso-value-loss lines. Finally, we show that a value loss of a single plan is measured by the amount of vertical deviation from the plan on the *VM* line.

The golden rule stock is defined as a point of maximal sustainable utility, where the terminal purified water stock must be as large as the initial stock. The level of the golden rule stock, at which the maximal utility is sustainable every period, is derived as a solution to the problem

$$\max u(x,x') \text{ subject to } x' \geq x \text{ for all } (x,x')\in\Omega. \tag{12}$$

Figure 2A shows the golden rule stock to be the unique one period plan $G = (\hat{x}, \hat{x})$ obtained by the intersection of the 45° line with *VM*. In this case, $\hat{x}$ can be derived as the solution to $\hat{x} = ((1+A)/a) - (1/a)\cdot\hat{x}$ which yields

$$\hat{x} = \frac{1+A}{1+a}. \tag{13}$$

Since $(\hat{x}, \hat{x})$ is the solution to the problem, appealing to Uzawa (1964)'s version of the Kuhn-Tucker theorem, there exists a golden rule price level $\hat{p}$ such that

$$u(x,x') + \hat{p}\cdot(x'-x) \leq u(\hat{x},\hat{x}) \text{ for all } (x,x')\in\Omega. \tag{14}$$

Proposition 1: *The golden rule price system is given by* $\hat{p} = \left(\dfrac{\hat{x}-A}{\dfrac{1}{a}-A}\right)$. Proof: included in Appendix.

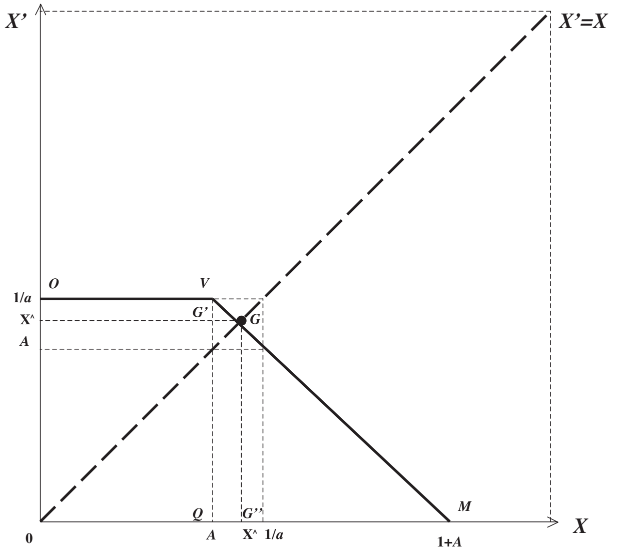
Then, we can define a value-loss such that (Figure 2B represents the iso-value-loss lines)

$$\delta_{[\hat{p},\hat{x}]}(x,x') = u(\hat{x},\hat{x}) - u(x,x') - \hat{p}(x'-x) \text{ for all } (x,x')\in\Omega. \tag{15}$$

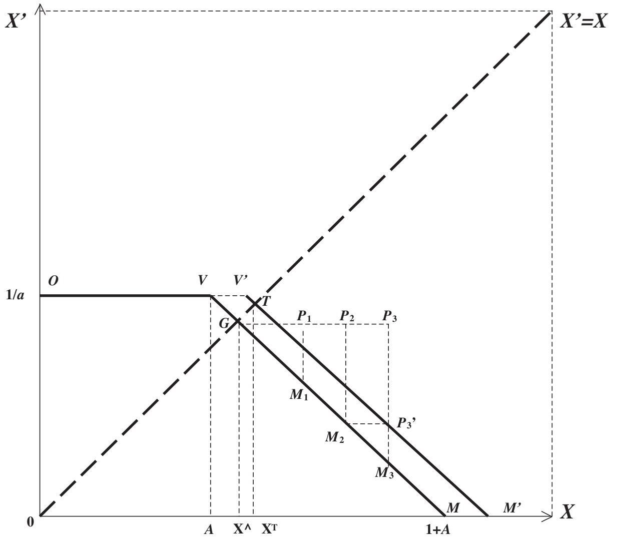
Proposition 2: *The VM line is the zero value-loss line, that is* $\delta_{[\hat{p},\hat{x}]}(x,x') = 0$ *for any* (x,x') *such that* $x' = \dfrac{1+A}{a} + \xi x$. Proof: included in Appendix.

Figure 2. Benchmarks

A.The golden rule stock



B. Iso-value-loss lines



We also show that even though the *OV* line is not a zero value-loss line, it is the line at which the value loss is minimized for an initial capital stock $x(0) \in (0, A)$.

Proposition 3: *For any initial stock* $x(0) \in (0, A)$, *any terminal plan where* $x' > (1/a)$ *is not feasible, and any plan with* $x' < (1/a)$ *has a higher value loss compared to one with* $x' = (1/a)$. Proof: included in Appendix.

Proposition 4: *A line parallel to VM, say V'M', is an iso-value-loss line, and the value loss of a plan on V'M' can be measured by the difference of the golden rule utility level and the utility level of a plan at which V'M' and the* 45° *line intersect*. Proof: included in Appendix.

Proposition 5: For *any plan* $(x, x') \in \Omega$ *such that* $x \in (A, 1 + A]$, *the more a plan is vertically deviated from the plan on the zero-value-loss line VM the more value loss it suffers*. Proof: included in Appendix.

Proposition 6: *The sum of the value losses of two plans equals the value loss of the sum of two plans*. Proof: included in Appendix.

F. Optimal policy

In this section, we present a complete characterization of optimal programs. An optimal program is one that minimizes the aggregate value loss and converges to the golden rule stock. Alternatively, any program that minimizes the aggregate of the sequence of all value losses over the long run is an optimal trajectory. Using cob web diagrams in today-tomorrow plane, every program starting from any initial capital stock can be tracked period by period, and its associated value loss per period and then its aggregate value loss can be calculated. In this way, we can compare the aggregate value losses of two different programs starting from the same initial capital stock and find an optimal program which has the minimum aggregate value loss. We consider two cases: when $a \leq 1$, or when $\xi \leq -1$. This is the case when the irrigation sector is more labor intensive than the purification sector. The other case is when $a > 1$, or when $\xi > -1$. This is the case when the purification sector is more labor intensive than the irrigation sector. Irrigation activities occur within the agricultural sector which is labor intensive, while water purification plants rely on water treatment machinery, which implies it is capital intensive. The model in this paper is, however, an extension of the Leontief two sector optimal growth model, which analyzes the optimal allocation of capital and labor to a consumption sector and an investment sector. The optimal policy, in this literature, is found to depend upon the factor intensity of the two sectors. Therefore, it is imperative to discuss all possible cases, even in this specific application.

Proposition 7: *If $\xi \leq -1$, an optimal program converges to a cycle. The size of the cycle depends on the value of ξ.* Proof: consider Figures 3, 4 and 5 below.

We have three possible scenarios. The first case is when $a = 1$, and $\xi = -1$, and thus $1+A = (1/a)+A$. As shown in Figure 3A, we have convergence to a cycle. If we start at any plan where $x(0) \in (0,A]$, we choose $x'=(1/a)$ at the OV line to minimize value loss. Then we end up in a nonconverging cycle of producing a stock of purified water of $(1/a)$ in one period and $[((1+A)/a)-(1/a)^2]=A$ in the other, and so on. Therefore, in this segment, we start with a stock of x which is used completely for drinking purposes, and all the labor is employed in purifying $(1/a)$ units of water. Then, in the second period a stock of $(1/a)$ is available, a portion A of it is used for drinking purposes, and the remainder $((1/a)-A)$ is used in irrigation to produce the consumption good, while the remaining labor $[1-((1/a)-A)]$ is employed to invest in water purification. So we end up with $z=(1/a)[1-((1/a)-A)]=[((1+A)/a)-(1/a)^2]=x'$. However, as $a = 1$, then $x'=[((1+A)/a)-(1/a)^2]=A$. Therefore, we start the third period with a stock of A. As this amount is used entirely for drinking purposes, all the labor is employed in purifying $(1/a)$ units of water. The subsequent periods rotate in the same nonconverging cycle thereafter.

The second case is if $1+A=(1/a)$. In this case, we have convergence to a cycle as well. The cycle, however is larger compared to the previous case. If we start at any plan where $x(0) \in (0,A]$, we choose $x'=(1/a)$ at the OV line to minimize value loss. Then we end up in a non-converging cycle of producing a stock of purified water of $(1/a)$ in one period and zero in the other, and so on, as shown in Figure 3B. Therefore, in this segment, we start with a stock of x which is used completely for drinking purposes, and all the labor is employed in purifying $(1/a)$ units of water. Then, in the second period a stock of $(1/a)=1+A$ is available, a portion A of it is used for drinking purposes, and the remainder 1 is used in irrigation to produce the consumption good, using all the available labor. Therefore, no labor is employed to invest in water purification. So we end up with $z=0=x'$. Therefore, we start the third period with a stock of zero. No water is used for drinking purposes, and all the labor is employed in purifying $(1/a)$ units of water. The subsequent periods rotate in the same non-converging cycle thereafter.

The remaining case is in between the previous two. If we start at any plan where $x(0)\in(0,A]$, we choose x'=(1/a) at the OV line to minimize value loss. This is because, as in Proposition 3, any terminal plan above (1/a) is not feasible, while any plan vertically downwards has a higher value loss. Then we end up in a non-converging cycle of producing a stock of purified water of (1/a) in one period and [((1+A)/a)-(1/a)^2] in the other, and so on, as shown in Figure 4. Therefore, in this first segment, we start with a stock of x which is used completely for drinking purposes, and all the labor is employed in purifying (1/a) units of water. Then, in the second period a stock of (1/a) is available, a portion A of it is used for drinking purposes, and the remainder ((1/a)-A) is used in irrigation to produce the consumption good, while the remaining labor [1-((1/a)-A)] is employed to invest in water purification. So we end up with z=(1/a)[1-((1/a)-A)]=[((1+A)/a)-(1/a)^2]=x'. Therefore, we start the third period with a stock of [((1+A)/a)-(1/a)^2]. As this amount is less than A, it is used entirely for drinking purposes, and all the labor is employed in purifying (1/a) units of water. The subsequent periods rotate in the same non-converging cycle thereafter.

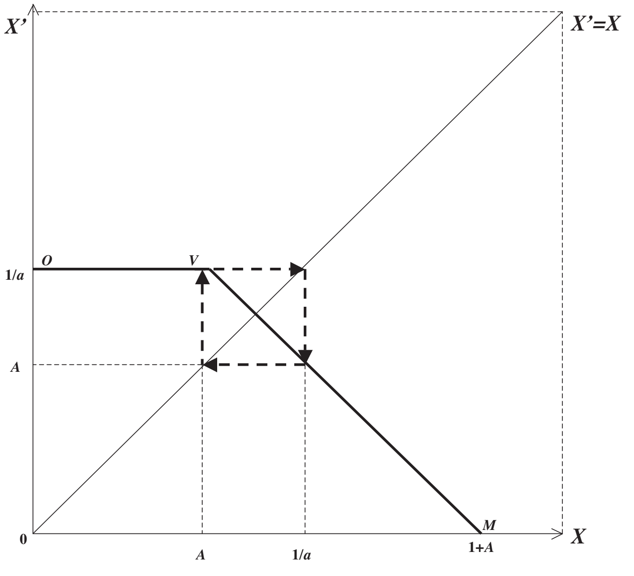
As shown in Figure 5, any plan in the second segment $x(0)\in[1+A-aA,1+A]$, we have x'=((1+A)/a)-(1/a)x, then we move to the 45° line where we start the cycle as we did with the first segment. The only segment left is where $x\in[A,1+A-aA]$. In this case, there are two subsegments. The first is if $x(0)\in[1+A-aA-a+a^2A,$ $1+A-aA]$, then you cycle out towards a sequence of purified water in the remaining segment $x\in(A,1+A-aA-a+a^2A)$, where plans in there end up in the previous segment and continue from there in a similar manner. Only if $x(0)=\hat{x}$, then there is convergence to the golden rule stock.

Proposition 8: *If* $\xi > -1$, *an optimal program from any initial stock converges to the golden rule stock in a damped cyclical way.* Proof: consider Figure 6.

In this case, if we start at any plan where $x(0) = A$, we have a damped cyclical convergence to the golden rule stock as shown in Figure 6. If we start at any plan where $x(0)\in(0,A]$, we choose x' = (1/a) at the OV line to minimize the value loss. Therefore, in this segment we start with a stock of x which is used completely for drinking purposes, and all the labor is employed in purifying (1/a) units of water. Then, in the second period a stock of (1/a) is available, a portion A of it is used for drinking purposes, and the remainder ((1/a)-A) is used in irrigation to produce the consumption good, while the remaining labor [1-((1/a)-A)] is employed to invest in water purification. So we end up with z=(1/a)[1-((1/a)-A)]=[((1+A)/a)-(1/a)^2]=x'.

Figure 3. Optimal paths. Convergence to an equilibrium cycle when $\xi \leq -1$

A. When 1+A = (1/a)+A (with a = 1, and ξ = -1)



B. When 1+A = 1/a (with ξ < -1)

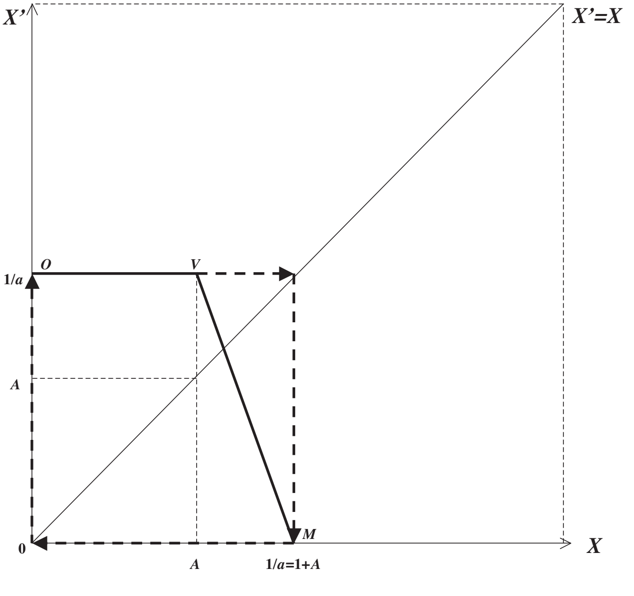


Figure 4. Convergence to an equilibrium cycle when $\xi \leq -1$

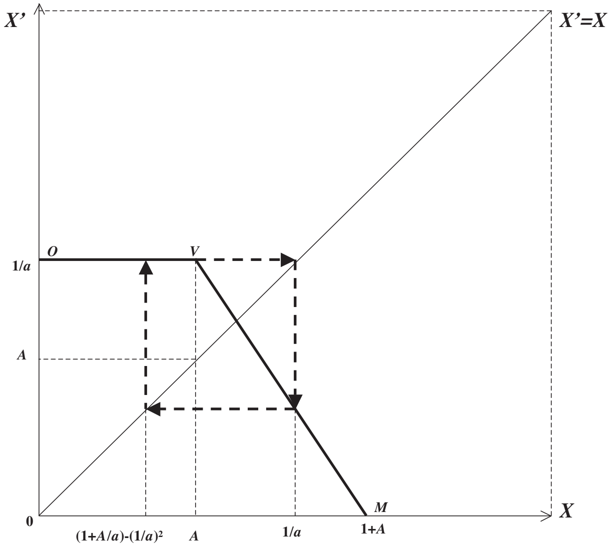


Figure 5. Segments and convergence

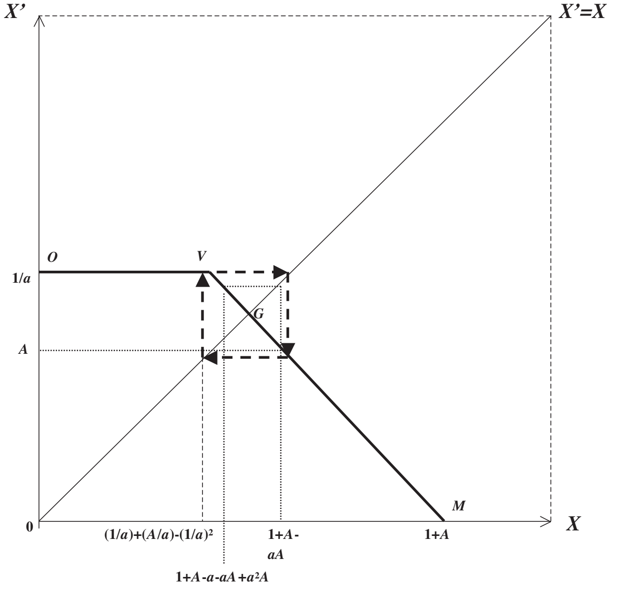
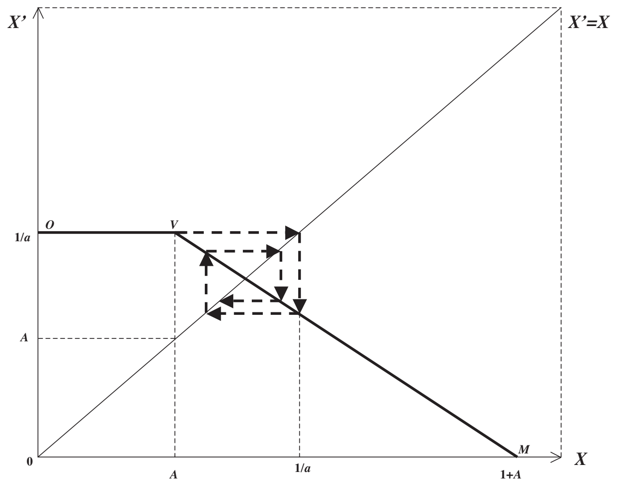


Figure 6. Damped cyclical convergence to the golden rule stock if $\xi > -1$



The difference from the case with $\xi \leq -1$ is that $[((1+A)/a)-(1/a)^2]>A$ in this case, while $[((1+A)/a)-(1/a)^2]<A$ in the previous case. Therefore, the optimal program converges to the golden rule stock in a damped cyclical way.

III. Conclusions

This paper uses a variant of the Leontief two sector optimal growth model, which analyzes the optimal allocation of capital and labor to a consumption sector and an investment sector in every period. The paper, however, attempts to extend the previous analysis in Fujio (2005, 2008) to the case of consumable capital; that is both the consumption good and capital can be consumed. Accordingly, a planner has preferences defined over both the consumption good and consumable capital. Future welfare levels are treated equally as current ones, that is the discount factor is assumed unity. The utility function with two arguments: the consumption good and the consumable capital, has implications on the planner's preferences, on the value-loss and accordingly on the optimal program.

An example of consumable capital is water resources. Water can be used in irrigation to produce an agricultural consumption good, and can also be used directly for drinking purposes. Therefore, water can serve as capital that is used in the production process of a consumption good, and can be consumable as well. In this context, the model features two sectors. The first utilizes labor to purify water.

Purified water can either be used for drinking purposes or in an irrigation sector along with labor to produce an agricultural consumption good. The geometric analysis characterizes the optimal path depending on whether the irrigation sector is more labor intensive than the purified sector. In this case, the optimal path is a non-converging cycle around the golden rule stock of purified water, while in the other case it exhibits a damped cyclical convergence to the golden rule stock.

This paper is not only an extension to a theoretical setup, but also an application to a specific case. The paper is motivated by a real world problem concerning the depletion of the scarce water resources due to the increasing world wide demand. In addition, the model assumptions are supported by observations from world data extracted from the Food and Agriculture Organization AQUASTAT. Finally, the model has policy implications, especially to developing countries that have significant agricultural and irrigation sectors, as it determines the optimal program to be adopted to ensure sustainability of water resources.

Appendix

Proof of Proposition 1. Substituting the zero consumption plan $V=(A,(1/a))$, and as the level of the consumption good is given by $(x\text{-}A)$ for all $x\in\left(A,1+A\right)$ we substitute $u(\hat{x},\hat{x})-u(A,1/a)=\left[(\hat{x}-A)-0\right]$ into equation (14) to obtain $\hat{x}-A\geq\hat{p}\cdot\left(\frac{1}{a}-A\right)$ and then $\hat{p}\leq\left(\frac{\hat{x}-A}{\frac{1}{a}-A}\right)$. Similarly, substituting the zero investment plan $M=(1+A,0)$, and $u(\hat{x},\hat{x})-u(1+A,0)=\left[(\hat{x}-A)-1\right]$ into equation (14), we have $\left(\hat{x}-A\right)-1\geq\hat{p}\cdot\left(0-\left(1+A\right)\right)$ and $\hat{p}\geq\frac{\left[\left(1+A\right)-\hat{x}\right]}{1+A}$. Therefore, $\left(\frac{\hat{x}-A}{\frac{1}{a}-A}\right)\geq\hat{p}\geq\frac{\left[\left(1+A\right)-\hat{x}\right]}{1+A}$. The weak inequalities are equalities in this expression because $\left(\frac{\hat{x}-A}{\frac{1}{a}-A}\right)=\frac{\left[\left(1+A\right)-\hat{x}\right]}{1+A}$ since $\left(\hat{x}-A\right)\cdot\left(1+A\right)=\left[\left(1+A\right)-\hat{x}\right]\cdot\left(\frac{1}{a}-A\right)$, and $\left(\hat{x}-A\right)\cdot\left(1+A\right)=\left[\left(1+A\right)-\hat{x}\right]\cdot\frac{1}{a}-\left[\left(1+A\right)-\hat{x}\right]\cdot A$, so $\left(\hat{x}-A\right)\cdot\left(1+A\right)=\hat{x}-\left[\left(1+A\right)-\hat{x}\right]\cdot A$ using the fact that $\hat{x}=\frac{1+A}{a}-\frac{1}{a}\cdot\hat{x}$.

Proof of Proposition 2. From Figure 2A, we can rewrite ξ using the golden rule price system as $\xi = -\frac{VQ}{QM} = -\frac{VG'}{G'G} = -\frac{\left(\frac{1}{a} - A\right) - (\hat{x} - A)}{(\hat{x} - A)} = -\left(\frac{\frac{1}{a} - \hat{x}}{\hat{x} - A}\right) = \left(\frac{\hat{x} - \frac{1}{a}}{\hat{x} - A}\right)$.

Therefore, we can write $\xi = \left(1 - \frac{1}{\hat{p}}\right) = \left[1 - \left(\frac{\frac{1}{a} - A}{\hat{x} - A}\right)\right] = \left(\frac{\hat{x} - \frac{1}{a}}{\hat{x} - A}\right)$. Similarly, we have that $\left(1 - \frac{1}{\hat{p}}\right) = \left[1 - \left(\frac{1 + A}{(1 + A) - \hat{x}}\right)\right] = \left(\frac{-\hat{x}}{(1 + A) - \hat{x}}\right) = -\frac{GG''}{G''M} = \xi$. Therefore, we obtain the equation of the *VM* line with some constant *D* as $x' = \left(1 - \frac{1}{\hat{p}}\right) \cdot x + D$. Since the *VM* line goes through the golden rule stock $G = (\hat{x}, \hat{x})$, we also have $\hat{x} = \left(1 - \frac{1}{\hat{p}}\right) \cdot \hat{x} + D$. Substituting one into the other yields: $\hat{x} = \hat{p} \cdot x' + x - \hat{p} \cdot x$.

We now compute the value loss of the *VM* line:

$$\delta_{(\hat{p},\hat{x})}(x, x') = u(\hat{x}, \hat{x}) - u(x, x') - \hat{p} \cdot (x' - x) = (\hat{x} - A) - (x - A) - \hat{p} \cdot x' + \hat{p} \cdot x$$
$$= \left[\hat{p} \cdot x' + x - \hat{p} \cdot x - A\right] - (x - A) - \hat{p} \cdot x' + \hat{p} \cdot x = 0.$$

Thus the *VM* line is the zero value-loss line.

Proof of Proposition 3. The *OV* line is not a zero value loss line. However for any initial stock in the segment $x \in (0, A)$, any terminal plan where $x' > (1/a)$ is not feasible. We can also show that any plan with $x' < (1/a)$ has a higher value loss compared to one with $x' = (1/a)$. We know that the value loss of any plan in this area is given by $\delta_{(\hat{p},\hat{x})}(x, x') = u(\hat{x}, \hat{x}) - u(x, x') - \hat{p} \cdot (x' - x)$. Comparing any plan $(x, (1/a))$ for all $x \in (0, A)$ to any other plan where $x' < (1/a)$, we find that they have a similar $u(\hat{x}, \hat{x}) - u(x, x')$ because at the two plans the planner consumes $w_p = x$ and $w_r = 0$. However, for the plan with a higher x', $\hat{p} \cdot (x' - x)$ is higher and thus it has a lower value loss compared to the one with lower x'. Therefore, any deviation vertically downwards from any point on the *OV* line increases the value loss. So, for any planner, if the current purified water stock is somewhere in the segment $x \in (0, A)$, the only option to minimize the value loss is to choose $x' = (1/a)$.

Proof of Proposition 4. We also show that the value loss of a plan on $V'M'$ in Figure 2B can be measured by the difference of the golden rule utility level and the utility level of a plan at which V′M′ and the 45º line intersects, such as $T=(x^T, x^T)$. The equation of $V'M'$ is obtained with some constant D such that $x'=\left(1-\frac{1}{\hat{p}}\right)\cdot x+D$. With the same procedure as before, we have $x^T=\left(1-\frac{1}{\hat{p}}\right)\cdot x^T+D$. Substituting one into the other yields $x^T=\hat{p}\cdot x'+x-\hat{p}\cdot x$. For any plan on $V'M'$, $c=(x-A)-\frac{(x'-z)}{(1-e)}$.

Therefore, $u(\hat{x},\hat{x})-u(x^T,x^T)=(\hat{x}-A)-(x^T-A)-\frac{(z-x')}{(1-e)}=(\hat{x}-A)-\hat{p}\cdot x'-x+\hat{p}\cdot x$ $+A-\frac{(z-x')}{(1-e)}$. For any plan on $V'M'$, the value loss is thus given by $\delta_{(\hat{p},\hat{x})}(x,x')$ $=u(\hat{x},\hat{x})-(x-A)-\frac{(z-x')}{(1-e)}-\hat{p}\cdot(x'-x)=u(\hat{x},\hat{x})-x+A+\hat{p}\cdot x-\hat{p}\cdot x'-\frac{(z-x')}{(1-e)}$ that is equal to $u(\hat{x},\hat{x})-u(x^T,x^T)$.

It is now clear that any plan on $V'M'$ has the same value loss. Since the utility level of a plan such as T is always less than the golden rule utility level, the value loss is always positive. If $V'M'$ shifts outwards from VM, then a point T gets far away from G which implies that the value loss increases. The same argument goes through for any plan in the triangle VQM.

Proof of Proposition 5. In the area VQM (Figure 2B), the value loss of a plan is given by $\delta_{(\hat{p},\hat{x})}(x,x')=u(\hat{x},\hat{x})-(x-A)-\hat{p}\cdot(x'-x)$. Then a marginal change in the value loss with respect to x' for any given x is $\frac{\partial\delta_{(\hat{p},\hat{x})}(x,x')}{\partial x'}=-\hat{p}<0$. Therefore, if a plan deviates vertically downwards from the plan on VM the value loss increases, and if it deviates by $\Delta x'<0$, the value loss associated with that plan is $-\Delta x'\cdot\hat{p}>0$. On the otherhand, above the VM line, $c=(x-A)+\frac{(z-x')}{(1-e)}$. So, the value loss of any plan in this area is given by $\delta_{(\hat{p},\hat{x})}(x,x')=u(\hat{x},\hat{x})-\left[(x-A)+\frac{(z-x')}{(1-e)}\right]-\hat{p}\cdot(x'-x)$.

Then, a marginal change in the value loss with respect to x' for any given x is given by $\frac{\partial \delta_{(\hat{p},\hat{x})}(x,x')}{\partial x'} = \frac{1}{1-e} - \hat{p} > 0$. If a plan deviates vertically upwards from the plan on VM, the value loss increases as well. If a plan deviates by $\Delta x' > 0$, the value loss associated with that plan is $\Delta x' \cdot ((1/(1-e)) - \hat{p}') > 0$.

Proof of Proposition 6. Pick up any two plans, say P_1 and P_2 on a horizontal line beginning at G in Figure 2B. The sum of these two plans is indicated by a plan P_3. Since ΔP_1GM_1 and $\Delta P_3'M_2M_3$ are congruent, the length of P_3M_3 is the sum of P_1M_1 and P_2M_2. This length stands for the magnitude of deviation from the zero-value-loss plan, which is $\Delta x'$. Then, a value loss of the sum of two plans are the sum of the value losses of two plans.

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