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Heterogeneity in Ability and Inheritance, Disparities, and Development

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ABSTRACT This paper investigates the impact of income inequality on economic growth. A two-period overlapping generations model is developed where agents are heterogeneous in innate abilities and inheritance. In the first period, they receive their inheritance and their abilities are revealed. There are only two types of abilities: high and low. Individuals decide on their education level, and divide their inheritance between spending on education and saving. In the second period, individuals supply their labor and allocate the labor income and the return to their saving between consumption and bequests to their offsprings. Initial capital stock is owned entirely by the capitalists. In this context, a more equal distribution of income enhances economic growth if the economy is lower than a threshold capital-labor ratio, while income inequality has an insignificant effect above this threshold. The predictions of the model are tested empirically using the Hansen (1999) threshold estimation. The results, using a panel of 70 countries for the period 1971-1999, suggest that there is a statistically significant threshold income per capita, below which the coefficient on the relationship between inequality and economic growth is significantly negative and above which the estimate is not significant.

KEY WORDS: Income inequality, economic growth
JEL CLASSIFICATIONS: D9, D31

1. Introduction

The distribution of income in an economy, and its effect on economic outcomes, has always been a source of concern for economists. In this context, there are two

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streams of literature. One argues that income inequality is propitious to economic performance, while the other concludes that the prevalent disparities call for an intervention to achieve the desired outcomes. According to Kuznets (1955), these attempts struggled in a 'field of study that has been plagued by looseness in definitions, unusual scarcity of data, and pressure of strongly held opinions'.

On one hand, a vast literature argues that greater egalitarian conditions are a prerequisite for economic growth, and that inequality adversely affects the overall performance of the economy. For instance, Barro (2000) shows that redistribution from the rich whose marginal productivity is low to the poor whose marginal productivity is high, but cannot invest in human capital more than their endowment due to capital market imperfections, would enhance productivity and growth.

From a political economy point of view, Persson and Tabellini (1994) show that in more inegalitarian economies a majority of voters prefer a higher level of redistribution, which reduces the incentives for investment. Alesina and Rodrick (1994) demonstrate that the more equitable the economy the better endowed the median voter with capital, and thus the lower the level of capital taxation and the higher is the economy's growth. Finally, Perotti (1992, 1993) emphasizes that income disparities motivate disruptive and destabilizing activities, and argues that redistributive policies reduce social tension.

Another stream of literature disputes the previous conclusions and asserts that the skewness of the income distribution is conducive to economic performance. In support of this opinion, Forbes (2000) provides empirical evidence that income inequality has a significant positive effect on economic growth. In this context, studies of consumption and saving behavior proposed a channel in which inequality has a stimulating effect on growth. For instance, Carroll (2000) finds that the marginal propensity to save of the rich is higher than that of the poor. This is because, according to the precautionary saving incentive, consumers with small asset stocks tend to compress their consumption so that their marginal propensity to consume out of wealth is higher than that of those holding larger asset stocks. The implication is that if the growth rate of income is proportional to aggregate saving, then more inegalitarian economies grow faster.

As an attempt to reconcile these two streams, Galor and Moav (2004) provide a unified approach that argues that, in the early stages of development, physical capital accumulation is the primary source of economic growth. Thus, inequality enhances growth by channeling resources towards individuals whose marginal propensity to save is higher. In later stages of development, physical capital is replaced by human capital as the engine of growth. Accordingly, equality alleviates the adverse effects of credit constraints on human capital accumulation and prompts the growth process.

This paper extends the framework in Galor and Moav (2004) by introducing agent heterogeneity across innate abilities and inheritance. This is to emphasize that what generates income inequality is not only differences in what one inherits but also differences in one's ability to generate income through labor. This extension suggests that a more equitable income distribution enhances economic growth at an earlier stage than the one proposed by Galor and Moav (2004).

An overlapping generations framework is developed in which individuals live for two periods. In the first period, they receive their inheritance and their innate abilities are revealed. The term inheritance is used interchangeably with bequests received from parents in this context. Individuals decide on their optimal education level, and divide their inheritance between spending on education and saving for the second period, and then devote their entire time to the acquisition of human capital. In the second period, individuals supply their efficiency units of labor and allocate the resulting labor income along with the return to their saving between consumption and bequests to their offsprings. Initial capital stock is entirely owned by a proportion of the population referred to as capitalists, while the remainder are referred to as workers. There are only two types of innate abilities: high and low. In this context, a more equitable distribution of income enhances economic growth if the economy is lower than a threshold capital-labor ratio, while income inequality has an insignificant effect above this threshold. The predictions of the model are tested empirically using the Hansen (1999) threshold estimation. The results, using a panel of 70 countries for the period 1971-1999, suggest that there is a statistically significant threshold income per capita, below which the coefficient on the relationship between inequality and economic growth is significantly negative and above which the estimate is not significant.

The remainder of the paper is organized as follows: section 2 includes the model, section 3 includes the empirical estimation, section 4 is the conclusion. Data and derivations are given in the appendices.

2. Model

Consider an overlapping generations economy in which economic activity extends over an infinite discrete time. In every period, a generation of measure 1 is born.

2.1 *Firms*

In every period, the economy produces a single homogeneous good. The factors of production are physical capital and human capital. The stock of physical capital in every period is the output produced in the preceding period net of consumption and human capital investment. The level of human capital in every period is a function of individuals' education decisions in the previous period. Output is produced at time t according to a neoclassical constant returns to scale production technology as follows:

$$Y_t = Y(K_t, H_t) = AH_t^{1-\alpha}K_t^{\alpha} \tag{1}$$

where A is the level of technology,[1] and $\alpha \in (0, 1)$. $K_t = \int_0^1 S_{it-1}di$ is the quantity of physical capital in efficiency units employed in production in period t, and is the sum of the saving S_{it-1} of all individuals in period $t-1$. $H_t = \int_0^1 H_{it-1}di$ is

[1]The introduction of technological progress that is fueled by human capital accumulation would not affect the qualitative results. If human capital accumulation is conducive for economic growth, the optimal evolution of the economy would require the fastest physical capital accumulation in early stages of development so as to raise the incentive to invest in human capital. Inequality in early stages would therefore stimulate the process of development.

the quantity of human capital in efficiency units employed in production in period t, and is the sum of the human capital acquired by all individuals in period $t-1$. Firms operate in a perfectly competitive environment. Given the wage rate per efficiency units of labor W_t and the rate of return to capital R_t, firms in period t choose the level of employment of physical and human capital so as to maximize their profits

$$\{K_t, H_t\} = \text{argmax}[AH_t^{1-\alpha}K_t^{\alpha} - W_tH_t - R_tK_t] \tag{2}$$

The firms demand the optimal level of physical capital, such that the return to physical capital is equal to its marginal productivity

$$R_t = A\alpha H_t^{1-\alpha}K_t^{\alpha-1} = A\alpha(k_t)^{\alpha-1} = R(k_t) \tag{3}$$

The firms demand the optimal level of human capital, such that the return to human capital is equal to its marginal productivity

$$W_t = A(1-\alpha)H_t^{-\alpha}K_t^{\alpha} = A(1-\alpha)(k_t)^{\alpha} = W(k_t) \tag{4}$$

where $k_t = \left(\frac{K_t}{H_t}\right)$ is the capital-labor ratio, $\frac{\partial R_t}{\partial k_t} = A\alpha(\alpha-1)k_t^{\alpha-2} < 0$, and $\frac{\partial W_t}{\partial k_t} = A(1-\alpha)\alpha k_t^{\alpha-1} > 0$. The first derivative ensures that physical capital is the engine for growth at an early stage of development when k_t is small. The last derivative emphasizes that the accumulation of physical capital increases the rate of return to human capital, and thus induces human capital accumulation.

2.2 *Individuals*

In every period, a generation that consists of a continuum of individuals of measure 1 is born. Each individual has a single parent and a single offspring. Individuals live for two periods. In the first period of their lives, they receive their inheritance and their abilities are revealed. For simplicity, we assume there are two types of innate abilities: high ability and low ability. Individuals decide on dividing their inheritance between spending on education and saving for the second period, and then devote their entire time to the acquisition of human capital. The consumption of an individual in the first period can be perceived as part of that of the parent. In the second period, individuals supply their efficiency units of labor and allocate the resulting labor income along with the return to their saving between consumption and bequests to their offsprings.

2.2.1 Second period. Individuals, within as well as across generations, are identical in their preferences. For individual i, preferences are represented by a log-linear utility function[2] over consumption during adulthood C_{it+1}, and the

[2]This formulation is common in the literature and suggests that individuals are ex-ante identical in their intertemporal preferences, although due to differences in income their marginal propensity to save may differ. This form is supported empirically by Altonji *et al.* (1997).

value of the bequest to the offspring B_{it+1} as follows:

$$(1-\beta)\log(C_{it+1}) + \beta \log(\theta + B_{it+1}) \tag{5}$$

where $\beta \in (0,1)$, and $\theta > 0$.

In the second period of life, individual i of generation t supplies the acquired efficiency units of labor H_{it+1}, at the competitive market wage W_{t+1}. In addition, the individual receives $R_{t+1}S_{it}$, which is the return to the first period savings after spending on education, where $R_{t+1} = 1 + r_{t+1} - \delta$, and the rate of depreciation $\delta = 1$ for simplicity. In this context, a member i of generation t maximizes his preferences subject to the following budget constraint:

$$C_{it+1} + B_{it+1} \leq W_{t+1}H_{it+1} + R_{t+1}S_{it} \tag{6}$$

A member i of generation t chooses the level of consumption expenditure, C_{it+1} such that:

$$\frac{1-\beta}{C_{it+1}} = \lambda_t \tag{7}$$

A member i of generation t chooses the level of bequests to leave to his offspring such that:

$$\frac{\beta}{\theta + B_{it+1}} = \lambda_t \tag{8}$$

where λ_t is the Lagrange multiplier. The first-order conditions can be combined to give the following optimal condition:

$$B_{it+1} = \frac{\beta}{1-\beta} C_{it+1} - \theta \tag{9}$$

Rearranging yields:

$$\begin{aligned} B_{it+1} &= \beta(C_{it+1} + B_{it+1}) - \theta(1-\beta) \\ &= \beta\left[(W_{t+1}H_{it+1} + R_{t+1}S_{it}) - \theta\left(\frac{1-\beta}{\beta}\right)\right] \end{aligned} \tag{10}$$

Therefore, the optimal bequest is given by:

$$B_{it+1} = \begin{cases} \beta\left[(W_{t+1}H_{it+1} + R_{t+1}S_{it}) - \theta\left(\frac{1-\beta}{\beta}\right)\right] & \text{if } (W_{t+1}H_{it+1} + R_{t+1}S_{it}) > \theta\left(\frac{1-\beta}{\beta}\right) \\ 0 & \text{if } (W_{t+1}H_{it+1} + R_{t+1}S_{it}) \leq \theta\left(\frac{1-\beta}{\beta}\right) \end{cases} \tag{11}$$

2.2.2 First period. The innate ability of each individual is revealed at the beginning of the first period of their lives, and they devote their entire time to the acquisition of human capital. The acquired level of human capital increases if their time investment is supplemented by capital investment. The bequests received from parents are allocated between the finance of education E_{it} and saving, thus $S_{it} = B_{it} - E_{it}$. We also assume that individuals cannot borrow to finance education.

As the indirect utility function is a strictly increasing function of the individuals' second period wealth, individual i chooses the optimal level of education L_{it} so as to maximize the second period wealth, $W_{t+1}H_{it+1} + R_{t+1}S_{it}$. This is subject to the efficiency units of labor that a member i of generation t supplies in the second period, which is a strictly increasing, strictly concave function of the individual's level of education in period t; $H_{it+1} = H(L_{it})$. In the absence of expenditure on education, individuals acquire one efficiency unit of labor as basic skills; $H(0) = 1$. This assumption is intended to allow those who do not receive bequests, and cannot spend on education, to receive labor income. This income increases with the increase in wages due to the increase in the capital-labor ratio until it crosses the threshold $((\theta(1-\beta))/\beta)$, after which they can leave bequests to their offsprings. Also assume that $\lim_{Lit\to 0} H'(L_{it}) = \gamma < \infty$, and $\lim_{Lit\to\infty} H'(L_{it}) = 0$. The expenditure on education is given by:

$$E_{it} = \Omega^j L_{it} \tag{12}$$

where Ω^j; $j \in (h, l)$ for high and low ability respectively, is the cost of a unit of education, such that $\Omega^h < \Omega^l$. This can be perceived as a reduction in the cost of education due to the ease with which high-ability individuals absorb the subject they are learning compared with low-ability ones. The optimal level of education is thus the solution to:

$$L_{it+1} = \text{argmax}[W_{t+1}H(L_{it}) + R_{t+1}(B_{it} - E_{it})] \tag{13}$$

The optimal level of education of individual i of generation t is chosen such that:

$$H'(L_{it}) = \frac{R_{t+1}}{W_{t+1}}\Omega^j = \frac{R(K_{t+1})}{W(k_{t+1})}\Omega^j \tag{14}$$

This means that the optimal level of education is unique and identical across those of the same ability category of generation t. Thus, the optimal education level for an individual i amongst those with high ability is given by $L_t^h = L^h(\Omega^h, k_{t+1})$, and for individual i amongst those with low ability is given by $L_t^l = L^l(\Omega^l, k_{t+1})$. Accordingly, the optimal expenditure on education is also identical across those of the same ability, such that $E_t^h = \Omega^h L_t^h$ and $E_t^l = \Omega^l L_t^l$.

Proposition 1 $\exists$ *a unique capital labor ratio* $k^j = \frac{\alpha}{(1-\alpha)\gamma}\Omega^j; j \in (h, l)$, *below which individuals with innate ability j do not invest in human capital, and such* *that for an individual i of this category:*

if $k_{t+1} \leq k^j$ *and if* $B_{it} = 0$, *then* $E_{it} = 0$ *and* $S_{it} = 0$,
while if $k_{t+1} \leq k^j$ *and if* $B_{it} > 0$, *then* $E_{it} = 0$ *and* $S_{it} = B_{it}$,
on the other hand, if $k_{t+1} > k^j$ *and if* $B_{it} = 0$, *then* $E_{it} = 0$ *and* $S_{it} = 0$,
while if $k_{t+1} > k^j$ *and if* $0 < B_{it} < E_t^j$, *then* $E_{it} = B_{it}$ *and* $S_{it} = 0$,
while if $k_{t+1} > k^j$ *and if* $B_{it} > 0$ *and* $B_{it} \geq E_t^j$, *then* $E_{it} = E_t^j$ *and* $S_{it} = B_{it} - E_t^j$.

Proof. The proof is included in the Appendix.

In this context, $k^h < k^l$ as $\Omega^h < \Omega^l$. If $k_{t+1} \leq k^h$, neither high nor low ability individuals invest in human capital. If $k^h < k_{t+1} \leq k^l$ only the high ability individuals invest in human capital. If $k_{t+1} > k^l$ all ability types invest in human capital.

2.3 *Aggregation*

Suppose that, in period $t = 0$, the economy consists of two groups of adult individuals: capitalists and workers, who differ only in their initial capital ownership. Capitalists are assumed a fraction σ of all adult individuals in the economy who equally own the entire initial physical capital in the economy, while workers are a fraction $(1 - \sigma)$ who have no ownership of the initial physical capital. We also assume that a proportion μ in the economy are high-ability individuals, while $(1 - \mu)$ are low-ability individuals. If it is equally likely to be high ability individual amongst capitalists or workers, then we have four categories: $\sigma\mu$ high-ability capitalists, $\sigma(1 - \mu)$ low-ability capitalists, $(1 - \sigma)\mu$ high-ability workers, and $(1 - \sigma)(1 - \mu)$ low-ability workers. Since individuals are homogeneous within a group, the uniqueness of the solution to their optimization problem assures that their offsprings are homogeneous as well. Therefore, the aggregate physical capital in period $t + 1$ is given by the sum of the savings of the four groups in period t as follows:

$$K_{t+1} = \sigma\mu S_t^{hc} + \sigma(1 - \mu)S_t^{lc} + (1 - \sigma)\mu S_t^{hw} + (1 - \sigma)(1 - \mu)S_t^{lw} \quad (15)$$

Similarly, aggregate human capital in period $t + 1$ is given by the sum of acquired human capital in period t as follows:

$$H_{t+1} = \sigma\mu H_t^{hc} + \sigma(1 - \mu)H_t^{lc} + (1 - \sigma)\mu H_t^{hw} + (1 - \sigma)(1 - \mu)H_t^{lw} \quad (16)$$

where the superscripts (hc, lc, hw, lw) are for high-ability capitalists, low-ability capitalists, high-ability workers and low-ability workers, respectively. In period $t = 0$, the level of human capital $H_0 = 1$, while the entire stock of physical capital $K_0 > 0$, is distributed equally between the capitalists. As the education expenditure of each group is a function of the capital-labor ratio k_{t+1} and bequests B_t of all types, the aggregate physical capital is given by $K_{t+1} = K(k_{t+1}, B_t^{hc}, B_t^{lc}, B_t^{hw}, B_t^{lw})$, and the aggregate human capital is given by $H_{t+1} = H(k_{t+1}, B_t^{hc}, B_t^{lc}, B_t^{hw}, B_t^{lw})$. Therefore, we can conclude that there exists a continuous single valued function, such that the capital-labor ratio in period $t+1$ is fully determined by the level of bequests of all types

$$k_{t+1} = k(B_t^{hc}, B_t^{lc}, B_t^{hw}, B_t^{lw}), \tag{17}$$

where $k_0 \in (0, k^h)$. This assumption ensures that in the initial stages the rate of return to physical capital is higher than the rate of return to human capital. The evolution of bequests within each group is given by:

$$B_{t+1}^{jc} = \max\left\{\beta\left[W_{t+1}H(L_t^{jc}) + R_{t+1}(B_t^{jc} - E_t^{jc}) - \theta\left(\frac{1-\beta}{\beta}\right)\right], 0\right\};\ j \in (h, l) \tag{18}$$

$$B_{t+1}^{jw} = \max\left\{\beta\left[W_{t+1}H(L_t^{jw}) + R_{t+1}(B_t^{jw} - E_t^{jw}) - \theta\left(\frac{1-\beta}{\beta}\right)\right], 0\right\};\ j \in (h, l) \tag{19}$$

where the initial bequests, at time $t = 0$, are given by:

$$\begin{aligned}
B_0^{hc} &= \max\left\{\beta\left[W(k_0) + R(k_0)\left(\frac{\mu}{\sigma}k_0\right) - \theta\left(\frac{1-\beta}{\beta}\right)\right], 0\right\}\\
B_0^{lc} &= \max\left\{\beta\left[W(k_0) + R(k_0)\left(\frac{1-\mu}{\sigma}\right)k_0 - \theta\left(\frac{1-\beta}{\beta}\right)\right], 0\right\}\\
B_0^{hw} &= \max\left\{\beta\left[W(k_0) - \theta\left(\frac{1-\beta}{\beta}\right)\right], 0\right\}\\
B_0^{lw} &= \max\left\{\beta\left[W(k_0) - \theta\left(\frac{1-\beta}{\beta}\right)\right], 0\right\}
\end{aligned} \tag{20}$$

2.4 *Stages of Development*

This section analyzes the endogenous evolution of the economy through different stages of development. The dynamical system is determined by the evolution of bequests of all types. The economy evolves through several stages as follows.

2.4.1 Stage I. Stage I is defined as the time interval $0 \le t < t^l$, where $t^l + 1$ is the first period in which the capital-labor ratio exceeds k^l. In this case, the income of the workers is less than the threshold that permits bequests.

Proposition 2 *In stage I, workers do not leave bequests to their offsprings.*

Proof. The proof is included in the Appendix.

In this stage, the capital-labor ratio is determined by the bequests of capitalists,

$$k_{t+1} = k(B_t^{hc}, B_t^{lc}, 0, 0) = \sigma\mu B_t^{hc} + \sigma(1-\mu)B_t^{lc} = \sigma B_t^c \text{ for } 0 < t < t^1,$$

and for $B_t^c \in [0, \mathrm{B}^l]$, where:

$$B^l = \left(\frac{k^l}{\sigma}\right) = \frac{\alpha}{(1-\alpha)\gamma\sigma}\Omega^1$$

The evolution of the economy in this stage is given by:

$$B_{t+1}^{jc} = \max\left\{\beta\left[W(\sigma B_t^c) + R(\sigma B_t^c)B_t^{jc} - \theta\left(\frac{1-\beta}{\beta}\right)\right], 0\right\};\ \textit{for } j \in (h, w)$$

$$B_{t+1}^{jw} = \max\left\{\beta\left[W(\sigma B_t^c) - \theta\left(\frac{1-\beta}{\beta}\right)\right], 0\right\};\ \textit{for } j \in (h, w) \tag{21}$$

In order to ensure that the economy ultimately takes off from stage I to stage II, we follow Galor and Moav (2004) in assuming that the technology is sufficiently productive.

Proposition 3 *The dynamical system in stage I has two steady state equilibria in the interval $B_t^c \in [0, B^l]$; a locally stable steady state, and an unstable steady state.*

Proof. The proof is included in the Appendix.

To ensure the system takes off from stage I to stage II, the initial bequest B_0^c is assumed between the unstable steady state and B^l, as in Figure 1. Workers are trapped in this stage in a zero bequest temporary steady state equilibrium. As the bequests of the capitalists increase, the capital-labor ratio increases, and the threshold bequest that enables the workers to escape the no-bequest temporary steady state eventually declines.

Proposition 4 *In stage I, if $k_{t+1} \in [k_0, k^h]$, a more equal income distribution does not enhance economic growth.*

Proof. The proof is included in the Appendix.

Proposition 5 *In stage I, if $k_{t+1} \in (k^h, k^l]$, redistribution from low-ability capitalists to high-ability workers enhances economic growth, if and only if $\frac{\beta}{\left(\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\right)\left(\frac{1}{\Omega^h}\right)} < \frac{\Omega^l}{\gamma}$.*

Proof. The proof is included in the Appendix.

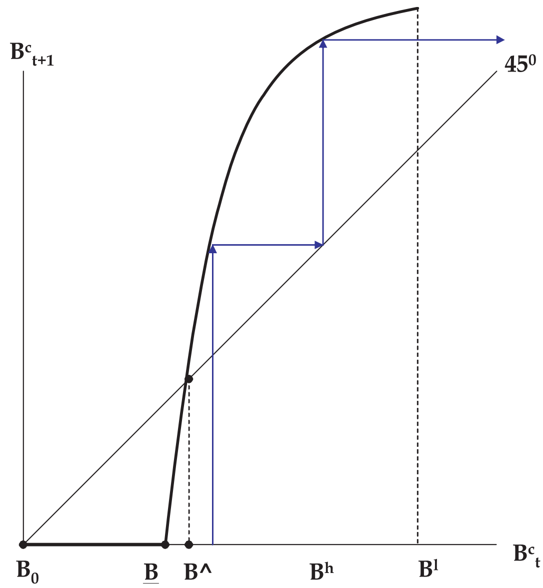

Figure 1. The dynamical system.

The term in the left-hand side of this inequality reflects the decline in the bequest received by the low-ability capitalists due to the transfer to the high ability workers that would allow them to increase their human capital by one unit exactly. The term $\frac{\Omega^l}{\gamma}$ reflects the required increase in education expenditure by those with low innate ability necessary to increase their human capital by one unit exactly if they start without any education. Accordingly, this condition states that it is cheaper to redistribute a small amount from the low ability capitalists to the high ability workers to increase human capital by one unit rather than attempting to accumulate human capital themselves.

This demonstrates that if this condition is satisfied, equality can be growth enhancing at an earlier stage of development than the one proposed by Galor and Moav (2004). In their study, equality can be growth enhancing only after the capital-labor ratio crosses the threshold k^l.

2.4.2 Stage II. Stage II is defined as the time interval $t^l \leq t \leq \bar{t}$, where $\bar{t}+1$ is the first period in which the capital-labor ratio exceeds $\bar{k}$, which is the critical level below which individuals who do not receive bequests from their parent do not leave bequests to their offspring. That is $W(\bar{k}) = \frac{\theta(1-\beta)}{\beta}$. The capital-labor ratio in the interval $(k^l, \bar{k}]$ is determined by the savings of the capitalists and their investment in human capital. Even though the marginal rate of return to human capital investment is higher than that for physical capital for the workers, since they are credit constraint as the economy is below $\bar{k}$, they do not save nor do they invest in human capital in this interval. Therefore,

$$k_{t+1} = \frac{\sigma\mu(B_t^{hc} - E_t^h) + \sigma(1-\mu)(B_t^{lc} - E_t^l)}{\sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma)}$$

There exists a unique value $\bar{B}$ such that $k_{t+1} = \bar{k}$. The evolution of the economy in this stage is given by:

$$B_{t+1}^{jc} = \beta\left[W(k_{t+1})H\left(\frac{E_t^j}{\Omega^j}\right) + R(k_{t+1})(B_t^{jc} - E_t^j) - \theta\left(\frac{1-\beta}{\beta}\right)\right]; \text{for } j \in (h, w)$$

$$B_{t+1}^{jw} = 0; \text{ for } j \in (h, w) \tag{22}$$

Proposition 6 *The dynamical system has no steady state equilibrium in the interval* $B_t^c \in [B^l, \bar{B}]$.

Proof. The proof is included in the Appendix.

Therefore, the bequests by the capitalists expand over the entire interval crossing into the next stage.

Proposition 7 *In stage II, a more equal income distribution enhances economic growth, if and only if* $\frac{\beta}{\left(\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\right)\left(\frac{1}{\Omega^h}\right) + \left(\frac{\partial H_t^{lw}}{\partial L_t^{lw}}\right)\left(\frac{1}{\Omega^l}\right)} < \frac{\Omega^l}{\gamma}$.

Proof. The proof is included in the Appendix.

The term on the left-hand side of this inequality reflects the decline in the bequest received by the low-ability capitalists due to the transfer to the high- and low-ability workers that would allow them to increase their human capital by one unit exactly. The term $\frac{\Omega^l}{\gamma}$ reflects the required increase in education expenditure by those with the low innate ability necessary to increase their human capital by one unit exactly if they start without any education. Accordingly, this condition states that it is cheaper to redistribute a small amount from the low ability capitalists to the high and low ability workers to increase human capital by one unit rather than attempting to accumulate human capital themselves.

2.4.3 Stage III. Stage III is defined as the time interval $\bar{t} < t < t^*$, where t^* is the time period in which the credit constraints are no longer binding for the workers. In this interval, the capital-labor ratio is given by:

$$k_{t+1} = \frac{\sigma\mu(B_t^{hc} - E_t^h) + \sigma(1-\mu)(B_t^{lc} - E_t^l)}{\sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma)\mu H^{hw}\left(\frac{B_t^{hw}}{\Omega^h}\right) + (1-\sigma)(1-\mu)H^{lw}\left(\frac{B_t^{lw}}{\Omega^l}\right)}$$

The evolution of the economy is given by:

$$B_{t+1}^{jc} = \beta \left[W(k_{t+1}) H \left(\frac{E_t^j}{\Omega^j} \right) + R(k_{t+1})(B_t^{jc} - E_t^j) - \theta \left(\frac{1-\beta}{\beta} \right) \right]; \text{ for } j \in (h, w)$$

$$B_{t+1}^{jw} = \max \left\{ \beta \left[W(k_{t+1}) H \left(\frac{B_t^{jw}}{\Omega^j} \right) - \theta \left(\frac{1-\beta}{\beta} \right) \right], 0 \right\}; \text{ for } j \in (h, w) \qquad (23)$$

Proposition 8 *The dynamical system has no steady state equilibrium in the time interval $\bar{t} < t < t^*$.*

Proof. The proof is included in the Appendix.

In the initial period $k_{\bar{t}+1} > \bar{k}$, the bequests $B_t^{hw} > 0$ and $B_t^{lw} > 0$, and therefore the sequence $\{B_t^c, B_t^w\}$ increases monotonically over the time period $\bar{t} < t < t^*$. Therefore, the economy proceeds into stage IV.

Proposition 9 *In stage III, a more equal income distribution enhances economic growth, if and only if* $\frac{\beta}{\left(\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\right)\left(\frac{1}{\Omega^h}\right)+\left(\frac{\partial H_t^{lw}}{\partial L_t^{lw}}\right)\left(\frac{1}{\Omega^l}\right)} < \frac{\Omega^l}{\gamma}$.

Proof. The proof is included in the Appendix.

2.4.4 Stage IV. Stage IV is defined as the time interval $t \geq t^*$, where credit constraints are no longer binding. The capital-labor ratio is given by:

$$k_{t+1} = \frac{\sigma\mu(B_t^{hc} - E_t^h) + \sigma(1-\mu)(B_t^{lc} - E_t^h) + (1-\sigma) \times \mu(B_t^{hw} - E_t^h) + (1-\sigma)(1-\mu)(B_t^{lw} - E_t^l)}{\sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma) \times \mu H^{hw}\left(\frac{E_t^h}{\Omega^h}\right) + (1-\sigma)(1-\mu)H^{lw}\left(\frac{E_t^l}{\Omega^l}\right)}$$

The evolution of the economy is given by:

$$B_{t+1}^{jc} = \beta \left[W(k_{t+1}) H \left(\frac{E_t^j}{\Omega^j} \right) + R(k_{t+1})(B_t^{jc} - E_t^j) - \theta \left(\frac{1-\beta}{\beta} \right) \right]; \text{ for } j \in (h, w)$$

$$B_{t+1}^{jw} = \beta \left[W(k_{t+1}) H \left(\frac{E_t^j}{\Omega^j} \right) + R(k_{t+1})(B_t^{jw} - E_t^j) - \theta \left(\frac{1-\beta}{\beta} \right) \right]; \text{ for } j \in (h, w) \qquad (24)$$

Proposition 10 *In Stage IV, income redistribution has no effect on economic growth.*

Proof. The proof is included in the Appendix.

3. Estimation

In this section, the predictions of the model are tested empirically using the threshold estimation technique developed in Hansen (1999). The model suggests that the relationship between economic growth and income inequality depends on the development stage of the economy. In the early stages, equality enhances economic growth, while in the later stage inequality does not have an effect on growth. The advantage of the threshold regression model is that it allows the level of GDP per capita to determine the existence and significance of a threshold level in the relationship between income inequality and economic growth, rather than imposing a priori an arbitrary classification scheme. The econometric specification is typical to that used in the literature to estimate the effect of income inequality on economic growth. The specification estimates the growth rate as a function of lagged income inequality, a lagged measure of human capital investment, and a lagged measure of physical capital investment. The threshold estimation model is, thus, given by:

$$Growth_{it} = \begin{cases} \mu_i + \beta_1 Gini_{it-1} + \phi_1 Education_{it-1} \\ \quad +\phi_2 Investment_{it-1} + e_{it} & \text{if } GDP_{it-1} \leq G^T \\ \mu_i + \beta_2 Gini_{it-1} + \phi_1 Education_{it-1} \\ \quad +\phi_2 Investment_{it-1} + e_{it} & \text{if } GDP_{it-1} > G^T \end{cases} \tag{25}$$

where the subscript i indexes the country, and the subscript t indexes time. The dependent variable $Growth_{it}$ denotes the growth rate of GDP per capita in country i in year t. The variable $Gini_{it-1}$ is a measure of the Gini coefficient in country i in year $t-1$. The variable $Education_{it-1}$ is a measure of educational attainment in country i in year $t-1$, and is considered as a proxy for human capital investment as is standard in the literature. The variable $Investment_{it-1}$ is the investment share of real GDP per capita in country i in year $t-1$. The variable GDP_{it-1} denotes real GDP per capita in country i in year $t-1$, and is the threshold variable determining the stage of development. It is standard in the literature to include initial real GDP per capita, as an independent variable, to test for convergence. If included, the equation contains a lagged endogenous variable which is the income term. This is apparent when the equation is rewritten with growth expressed as the difference in the logarithm of income levels. As Hansen's (1999) technique is developed for a non-dynamic panel, lagged real GDP per capita is excluded from this regression.

Obviously, the threshold GDP per capita determines whether the coefficient on the Gini coefficient is positive or negative. In this context, the observations are divided into two regimes depending on whether the threshold variable GDP_{it-1} is smaller or larger than the threshold G^T. The regimes are distinguished by

Table 1. Summary statistics

	Minimum	25% quantile	Median	75% quantile	Maximum
$Growth_{it}$	−1.004230	−0.002482	0.021078	0.043438	0.463332
$Education_{it}$	0.063400	0.750200	1.474800	2.396000	5.742000
$FemaleEducation_{it}$	0.201600	1.112000	2.248000	1.221000	1.494600
$MaleEducation_{it}$	0.580800	1.251000	1.712400	1.462000	2.028600
$Gini_{it}$	24.069090	36.424790	42.397830	47.224380	58.975360
GDP_{it}	474.417775	3359.168884	6322.430258	15183.425690	64336.281600
$Investment_{it}$	0.141677	9.970844	16.721896	22.481970	52.530579

differing regression slopes, β_1 and β_2. According to the predictions of the model, the coefficient β_1 is expected to be negative, while the coefficient β_2 is not expected to be statistically significant. Another way of writing the equation of interest is:

$$Growth_{it} = \mu_i + \beta_1 Gini_{it-1} I(GDP_{it-1} \leq G^T) + \beta_2 Gini_{it-1} I(GDP_{it-1} > G^T) + \phi_1 Education_{it-1} + \phi_2 Investment_{it-1} + e_{it} \tag{26}$$

where $I(.)$ is the indicator function. A balanced panel annual data is used for 70 countries that cover the period from 1971 to 1999. A Gini coefficient compiled by the University of Texas Inequality Project is used as a proxy for income inequality. The average years of total education in the population aged over 15, from Barro and Lee data on educational attainment, is used as a measure of human capital. Finally, real GDP per capita, and the investment share of real GDP per capita are extracted from the Penn World Tables 6.2. Detailed data description is included in the Appendix. Summary statistics of the variables used in the estimation are provided in Table 1.

To determine the number of thresholds, the model is estimated by least squares allowing for zero, one, two and three thresholds. The test statistics F_1, F_2, and F_3, along with their bootstrap[3] p-values are shown in column 1 in Table 2. The test for a single threshold F_1 is highly significant with a bootstrap p-value of zero, and the test for a double threshold F_2 is also significant with a bootstrap p-value of 0.006667. On the other hand, the test for a triple threshold F_3 is not significant, with a bootstrap p-value of 0.573333. Thus, we conclude that there is evidence that there are two thresholds in the regression relationship. For the remainder of the analysis, we work with the double threshold model as follows:

$$\begin{aligned} Growth_{it} = \mu_i &+ \beta_1 Gini_{it-1} I(GDP_{it-1} \leq G^{T1}) \\ &+ \beta_2 Gini_{it-1} I(G^{T1} < GDP_{it-1} \leq G^{T2}) \\ &+ \beta_3 Gini_{it-1} I(G^{T2} < GDP_{it-1}) + \phi_1 Education_{it-1} \\ &+ \phi_2 Investment_{it-1} + e_{it} \end{aligned} \tag{27}$$

We refer to this as regression 1. We also estimate another model, where we replace total educational attainment with male and female educational attainment. The

[3] 300 bootstrap replications are used for each of the three bootstrap tests.

Table 2. Tests for thresholds effects

	Regression 1	Regression 2
Test for single threshold		
F_1	87.368146	87.239120
P-value	0.000000	0.000000
(10%, 5%, 1% *critical values*)	(21.482757, 30.922510, 51.904671)	(20.606181, 30.059235, 51.775242)
Test for double threshold		
F_2	43.935402	46.205005
P-value	0.006667	0.006667
(10%, 5%, 1% *critical values*)	(22.111406, 27.344110, 39.992911)	(22.501746, 27.741156, 39.195563)
Test for triple threshold		
F_3	8.447935	7.772251
P-value	0.573333	0.636667
(10%, 5%, 1% *critical values*)	(16.833437, 19.644674, 28.538617)	(17.858530, 20.392893, 28.352943)

variable $MaleEducation_{it-1}$ is a measure of male educational attainment in country i in year $t-1$, and the variable $FemaleEducation_{it-1}$ is a measure of female educational attainment in country i in year $t-1$. The average years of education in the male and female population aged over 15 is extracted from Barro and Lee data on educational attainment. The test statistics F_1, F_2, and F_3, along with their

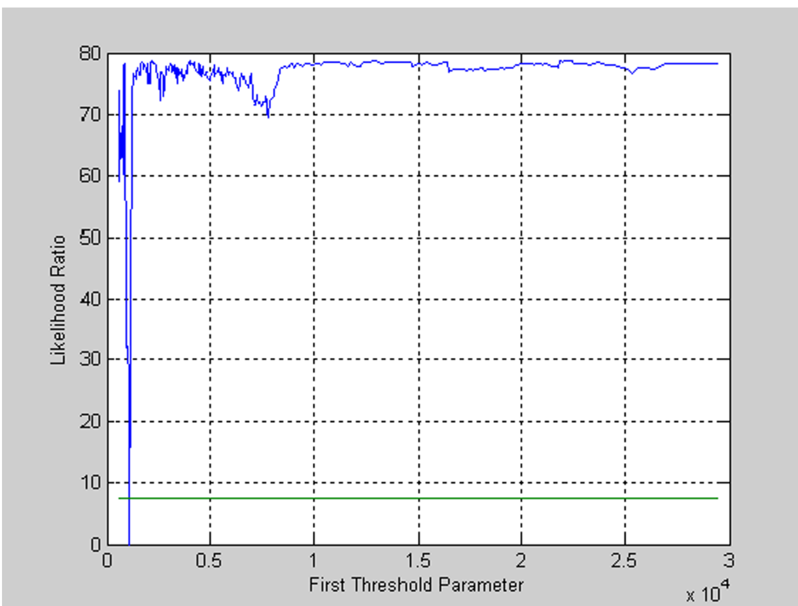

Figure 2. Confidence interval construction in double threshold model.

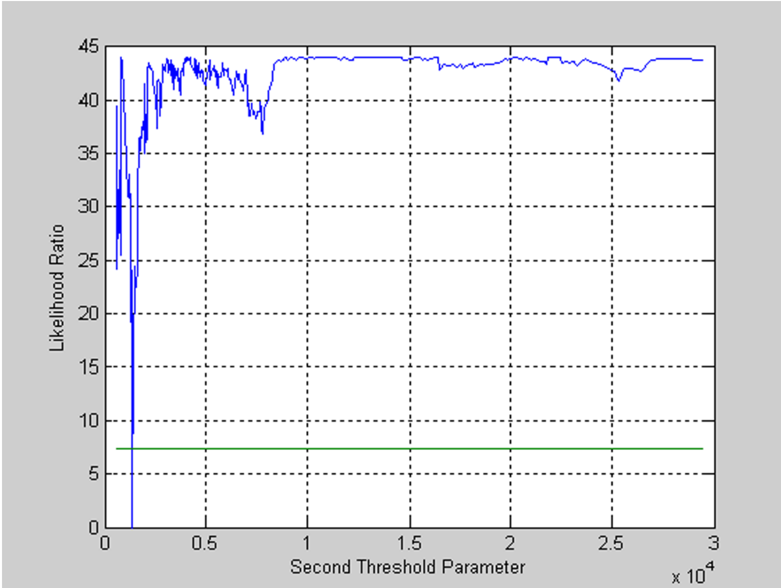

Figure 3. Confidence interval construction in double threshold model.

bootstrap[4] p-values are shown in column 2 in Table 2. We can conclude that there are also two thresholds in the regression relationship 2. Regression 2 is, thus, given by:

$$\begin{aligned} Growth_{it} = \mu_i &+ \beta_1 Gini_{it-1} I(GDP_{it-1} \leq G^{T1}) \\ &+ \beta_2 Gini_{it-1} I(G^{T1} < GDP_{it-1} \leq G^{T2}) \\ &+ \beta_3 Gini_{it-1} I(G^{T2} < GDP_{it-1}) + \phi_1 MaleEducation_{it-1} \\ &+ \phi_2 FemaleEducation_{it-1} + \phi_3 Investment_{it-1} + e_{it} \end{aligned} \tag{28}$$

In both regressions, the point estimates of the two thresholds are \$1079.172759 and \$1347.094572, and their asymptotic 99% confidence intervals are [1079.172759, 1079.172759] and [1347.094572, 1380.328760], respectively. More information can be learned from plots of the concentrated likelihood ratio function displayed in Figures 2 and 3. To examine the first-step likelihood ratio function which is computed when estimating a single threshold model, the first-step threshold estimate is the point where the likelihood function equals zero, which occurs at $G^{T1} = 1079.172759$. There is a second dip in the likelihood ratio around the second-step estimate $G^{T2} = 1347.094572$.

The regression slope estimates, conventional OLS standard errors, and white-correlated standard errors are reported in Table 3 for regression 1, and in Table 4 for regression 2. The estimates of primary interest in regression 1 are those on the Gini coefficient. Income inequality has a statistically significant negative effect on

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[4]300 bootstrap replications are used for each of the three bootstrap tests.

Table 3. Regression 1 estimates

Regressor	Coefficient estimate	OLS SE	White SE
$Education_{it-1}$	−0.005887	0.003488	0.003381
$Investment_{it-1}$	0.000986	0.000353	0.000651
$Gini_{it-1}I(GDP_{it-1} \leq 1079.172759)$	−0.004319	0.000711	0.002403
$Gini_{it-1}I(1079.172759 < GDP_{it-1} \leq 1347.094572)$	−0.000688	0.000599	0.000956
$Gini_{it-1}I(1347.094572 < GDP_{it-1})$	0.001018	0.000545	0.000622

Table 4. Regression 2 estimates

Regressor	Coefficient estimate	OLS SE	White SE
$MaleEducation_{it-1}$	−0.022579	0.011265	0.009896
$FemaleEducation_{it-1}$	0.016378	0.011019	0.009469
$Investment_{it-1}$	0.000966	0.000353	0.000650
$Gini_{it-1}I(GDP_{it-1} \leq 1079.172759)$	−0.004395	0.000711	0.002411
$Gini_{it-1}I(1079.172759 < GDP_{it-1} \leq 1347.094572)$	−0.000778	0.000601	0.000966
$Gini_{it-1}I(1347.094572 < GDP_{it-1})$	0.000982	0.000545	0.000623

economic growth with a coefficient of −0.004319, if real GDP per capita is below the first threshold. The coefficient is negative, but not statistically significant, if real GDP per capita is between the first and the second thresholds. Finally, income inequality has an insignificant impact on economic growth, if real GDP per capita is above the second threshold.

In regression 2, income inequality has a statistically significant negative effect on economic growth with a coefficient of −0.004395, if real GDP per capita is below the first threshold. The coefficient is also negative, but not statistically significant, if real GDP per capita is between the first and the second thresholds. Finally, income inequality has an insignificant impact on economic growth, if real GDP per capita is above the second threshold.

4. Conclusion

Theoretical proposals and empirical estimations have provided contradictory conclusions as to whether income inequality is propitious to economic performance, or whether it acts as an impediment to growth. Galor and Moav (2004) provide a reconciliation, and argue that the replacement of physical capital accumulation by human capital accumulation as the prime engine for economic growth changes the impact of inequality on growth. In the early stage of development, inequality enhances the process of development by channeling resources towards the owners of capital whose marginal propensity to save is higher, while in later stages equality alleviates credit constraints on the investment in human capital and promotes economic growth.

This paper extends the Galor and Moav (2004) model by introducing heterogeneity in innate abilities and inheritance. An overlapping generations framework is developed in which individuals live for two periods. In the first period, they receive their inheritance and their innate abilities are revealed. Individuals decide on their optimal education level, and divide their inheritance between spending on education and saving for the second period, and then devote their entire time to the acquisition of human capital. In the second period, individuals supply their efficiency units of labor and allocate the resulting labor income along with the return to their saving between consumption and bequests to their offsprings. Initial capital stock is owned entirely by capitalists, while the remainder of the population are referred to as workers. There are only two types of innate abilities: high and low.

In this context, a more equitable distribution of income enhances economic growth if the economy is lower than a threshold capital-labor ratio, while income inequality has an insignificant effect above this threshold. The predictions of the model are tested empirically using the Hansen (1999) threshold estimation. The results, using a panel of 70 countries for the period 1971–1999, suggest that there is a statistically significant threshold income per capita, below which the coefficient on the relationship between inequality and economic growth is significantly negative and above which the estimate is not significant.

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Appendix

Data

The estimation uses annual data that covers the period from 1971 to 1999 for 70 countries, namely: Algeria, Australia, Austria, Bangladesh, Barbados, Belgium, Bolivia, Cameroon, Canada, Central Africa, Chile, Colombia, Cyprus, Denmark, Dominican Republic, Ecuador, El Salvador, Fiji, Finland, Germany, Ghana, Greece, Guatemala, Haiti, Hong Kong, Hungary, Iceland, India, Indonesia, Iran, Iraq, Ireland, Israel, Italy, Jamaica, Japan, Jordan, Kenya, Korea, Kuwait, Malawi, Malaysia, Mauritius, Mexico, Netherlands, New Zealand, Nicaragua, Norway, Pakistan, Panama, Papua New Guinea, Philippines, Poland, Portugal, Senegal, Singapore, South Africa, Spain, Swaziland, Sweden, Syria, Taiwan, Tanzania, Tunisia, Turkey, United Kingdom, United States of America, Uruguay, Venezuela, and Zimbabwe.

The variables used in the estimations are described in details as follows.

Gini Coefficient. A detailed description of the Estimated Household Income Inequality dataset, compiled by the University of Texas Inequality Project, is provided in Galbraith and Kum (2004). This dataset combines data on the measure of dispersion of pay across industrial categories in the manufacturing sector, drawn from the industrial database published annually by the United Nations Industrial Development Organization UNIDO, with the information in the Deininger and Squire (1996) data, resulting in a dataset for the Gini coefficient referred to as the Estimated Household Income Inequality EHII.

Gross Domestic Product. The data for real Gross Domestic Product per capita (Laspeyres) is extracted from the Penn World Tables 6.2, which is obtained by adding up consumption, investment, government and exports, and subtracting imports in any given year, where the components are obtained by extrapolating the 1996 values in international dollars from the Geary aggregation using national growth rates. The growth rate of GDP per capita is given by the difference in the natural logarithm of the real GDP per capita in two consecutive years.

Education. The data for total education, male and female education are derived from the Barro and Lee International Data on Educational Attainment in which they constructed estimates of educational attainment by sex for persons aged 15 and over. The values applied to several countries over five-year intervals for 1970, 1975, 1980, 1985, 1990, 1995 and 1999. The estimation procedure began with census information on school attainment for males and females where the data came from individual governments as compiled by the UNESCO and other sources. We follow Forbes (2000) in using the average years of secondary schooling in the male population, and the average years of secondary schooling in the female population as proxies for male and female education, respectively. As the data are available only for the years 1970, 1975, 1980, 1985, 1990, 1995 and 1999, we use linear interpolation to derive the years in between.

Investment. The data for the investment share of real Gross Domestic Product per capita is extracted from the Penn World Tables 6.2.

Derivations

Proof of Proposition 1 As $\lim_{L_{it}\to 0} H'(L_{it}) = \gamma$, then the first order condition for those with innate ability j satisfies $H'(L_{it}) = \frac{R(k^j)}{W(k^j)}\Omega^j = \gamma$. Therefore, from the first order conditions of the firm $\frac{A\alpha(k^j)^{\alpha-1}}{A(1-\alpha)(k^j)^{\alpha}}\Omega^j = \gamma$, which can be rearranged as $k^j = \Omega^j \frac{\alpha}{(1-\alpha)\gamma}$.

If $k_{t+1} \leq k^j$, then individual i of generation t with innate ability j does not invest in education $E_{it} = 0$, and saves only if he receives a positive bequest from his parents $B_{it} > 0$, since $S_{it} = B_{it} - E_{it}$. On the other hand, if $k_{t+1} > k^j$, individual i of generation t with innate ability j invests in education where the optimal level is given by E_t^j. However, if they do not receive any bequest from their parents, they are credit constrained since they cannot borrow to cover the expenditure on education, and thus they do not spend on education $E_{it} = 0$ and save nothing $S_{it} = 0$. If they receive a positive bequest from their parents $B_{it} > 0$, and this amount is lower than the optimal level E_t^j, then their expenditure on education is constrained by this amount and they do not save as well. They can only achieve the optimal level of education if the bequest they receive from their parents is equal to or larger than the optimal amount. If larger, they save the difference $S_{it} = B_{it} - E_t^j$.

Proof of Proposition 2 Let $\bar{k}$ be the critical level of the capital-labor ratio below which individuals who do not receive bequests from their parent do not leave bequests to their offspring. That is $W(\bar{k}) = \frac{\theta(1-\beta)}{\beta}$. From the first-order conditions of the firm, $\bar{k} = \left[\frac{\frac{\theta(1-\beta)}{\beta}}{A(1-\alpha)}\right]^{\frac{1}{\alpha}}$, where if $k_{t+1} \leq \bar{k}$ then $W(k_{t+1}) \leq \frac{\theta(1-\beta)}{\beta}$, whereas if $k_{t+1} > \bar{k}$, then $W(k_{t+1}) > \frac{\theta(1-\beta)}{\beta}$. Accordingly, $B_{it+1} = 0$ if $k_{t+1} \leq \bar{k}$, and $B_{it+1} > 0$ if $k_{t+1} > \bar{k}$.

Assume also that once wages increase sufficiently such that workers leave bequests to their offspring, or if $k_{t+1} > \bar{k}$, investment in human capital is profitable for all types of workers, or $k_{t+1} > k^l$. That is $k^l \leq \bar{k}$. Since $\frac{\partial \bar{k}}{\partial(\frac{\theta(1-\beta)}{\beta})} > 0$, it follows that for any given γ, there exists $\frac{\theta(1-\beta)}{\beta}$ sufficiently large such that $k^l \leq \bar{k}$.

Let $t^l + 1$ be the first period in which the capital-labor ratio exceeds k^l. That is, since $k_0 < k^l$, it follows that $k_{t+1} \leq k^l$ for all $0 \leq t < t^l$. Let $\bar{t} + 1$ be the first period in which the capital-labor ratio exceeds $\bar{k}$. That is $k_{t+1} \leq \bar{k}$ for all $0 \leq t < \bar{t}$. Then, it follows from $k^l \leq \bar{k}$ that $t^l \leq \bar{t}$.

Since $k_0 < k^l$, then the bequest of a worker $B_0^w = 0$. Furthermore, for $1 \leq t < \bar{t}$, as long as $B_{t-1}^w = 0$, the descendants do not invest in human capital, $H_t^w = 1$, and therefore $B_t^w = \max[\beta(W(k_{t+1}) - \frac{\theta(1-\beta)}{\beta}), 0] = 0$.

Proof of Proposition 3 There exists a $\underline{B}$ such that $B_{t+1}^c = 0$ for $B_t^c \leq \underline{B}$, where $B_t^c = \mu B_t^{hc} + (1-\mu)B_t^{lc}$. Thus, $\underline{B}$ is defined, such that:

$$0 = \beta\left[W(\sigma\underline{B}) + R(\sigma\underline{B})\underline{B} - \theta\frac{1-\beta}{\beta}\right]$$

$$0 = \beta\left\{(1-\alpha)A[\sigma\underline{B}]^\alpha + \alpha A[\sigma\underline{B}]^{\alpha-1}\underline{B} - \theta\left(\frac{1-\beta}{\beta}\right)\right\}$$

Solving for $\underline{B}$ yields $\underline{B} = [\frac{(\theta\frac{1-\beta}{\beta})}{(1-\alpha)A\sigma^\alpha + \alpha A\sigma^{\alpha-1}}]^{\frac{1}{\alpha}}$. Since this expression is decreasing in A, and we assumed the technology is sufficiently productive, then $B_{t+1}^c = 0$ for $B_t^c \leq \underline{B}$ as shown in Figure 1. Furthermore, B_{t+1}^c is increasing and strictly concave in the interval $B_t^c \in (\underline{B}, B^l]$, where $B^l = (\frac{k^l}{\sigma}) = \frac{\alpha}{(1-\alpha)\gamma\sigma}\Omega^l$, which is independent of A. As depicted in Figure 1, the function B_{t+1}^c is equal to zero for $B_t^c \leq \underline{B}$, and it is increasing and concave for $\underline{B} < B_t^c \leq B^l$, and it crosses the 45° line once in this interval. It is worth mentioning that in this interval, there is also $B^h = (\frac{k^h}{\sigma}) = \frac{\alpha}{(1-\alpha)\gamma\sigma}\Omega^h < B^l$, as $\Omega^h < \Omega^l$. Hence, the dynamical system has two steady state equilibria in the interval $B_t^c \in [0, B^l]$; a locally stable steady state, $B_0 = 0$, and an unstable steady state $\hat{B}$. If $B_t^c < \hat{B}$, the bequests contract over time and the system converges to the steady state B_0. If $B_t^c > \overset{\frown}{B}$, the bequests expand over the entire interval crossing into stage II. To ensure that the process of development starts in stage *I* and ultimately takes off to stage II, it is assumed that $B_0^c \in (\hat{B}, B^l)$.

Proof of Proposition 4 In stage I, if $k_{t+1} \in [k_0, k^h]$, we know that $B_t^{hc} > 0$, and $B_t^{lc} > 0$, while from Proposition 2, $B_t^{hw} = B_t^{lw} = 0$. From Proposition 1, high ability capitalists do not invest in education, so $E_t^{hc} = 0$, and thus $S_t^{hc} = B_t^{hc}$. Similarly, low ability capitalists do not invest in education, so $E_t^{lc} = 0$, and thus $S_t^{lc} = B_t^{lc}$. From Proposition 1, high ability workers do not invest in education, therefore $E_t^{hw} = 0$ and $S_t^{hw} = 0$. Similarly, low-ability workers do not invest in education, therefore $E_t^{lw} = 0$ and $S_t^{lw} = 0$. In this context, the aggregate physical capital is given by $K_{t+1} = \sigma\mu B_t^{hc} + \sigma(1-\mu)B_t^{lc}$, while the aggregate human capital is given by $H_{t+1} = 1$. Therefore, output in period $t+1$ is given by $Y_{t+1} = A[\sigma\mu B_t^{hc} + \sigma(1-\mu)B_t^{lc}]^\alpha$. In this context, we have:

$$B_t^{jc} = \beta\left(W_t + R_t B_{t-1}^{jc} - \frac{\theta(1-\beta)}{\beta}\right) = \beta\left(I_t^{jc} - \frac{\theta(1-\beta)}{\beta}\right);\ \text{for } j \in (h, l)$$

If ε_t, that is sufficiently small, is subtracted from the period t income of the capitalists, such that $\bar{I}_t^{hc} = I_t^{hc} - \mu\varepsilon_t$ and $\bar{I}_t^{lc} = I_t^{lc} - (1-\mu)\varepsilon_t$, and redistributed to workers such that $\bar{I}_t^{hw} = I_t^{hw} + \mu\varepsilon_t$ and $\bar{I}_t^{lw} = I_t^{lw} + (1-\mu)\varepsilon_t$. Then, we can see that $\frac{\partial B_t^{hc}}{\partial\varepsilon_t} < 0$ and $\frac{\partial B_t^{lc}}{\partial\varepsilon_t} < 0$. In addition, as long as $\bar{I}_t^{hw} < \frac{\theta(1-\beta)}{\beta}$ and $\bar{I}_t^{lw} < \frac{\theta(1-\beta)}{\beta}$, then $\frac{\partial B_t^{hw}}{\partial\varepsilon_t} = 0$ and $\frac{\partial B_t^{lw}}{\partial\varepsilon_t} = 0$. Accordingly, the effect of redistribution on output in period $t+1$ is given by:

$$\frac{\partial Y_{t+1}}{\partial\varepsilon_t} = \frac{\partial Y_{t+1}}{\partial K_{t+1}}\frac{\partial K_{t+1}}{\partial B_t^{hc}}\frac{\partial B_t^{hc}}{\partial\varepsilon_t} + \frac{\partial Y_{t+1}}{\partial K_{t+1}}\frac{\partial K_{t+1}}{\partial B_t^{lc}}\frac{\partial B_t^{lc}}{\partial\varepsilon_t} < 0$$

To show that $Y_{t+1} = Y(Y_t)$, we have:

$$\begin{aligned} Y_{t+1} &= A\{\sigma\mu[\beta W_t + \beta R_t B_{t-1}^{hc} - \theta(1-\beta)] \\ &\quad + \sigma(1-\mu)[\beta W_t + \beta R_t B_{t-1}^{lc} - \theta(1-\beta)]\}^\alpha \\ &= A\{\sigma\beta W_t - \sigma\theta(1-\beta) + \beta R_t[\sigma\mu B_{t-1}^{hc} + \sigma(1-\mu)B_{t-1}^{lc}]\}^\alpha \\ &= A\left\{\sigma\beta W_t - \sigma\theta(1-\beta) + \beta R_t\left[\left(\frac{Y_t}{A}\right)^{\frac{1}{\alpha}}\right]\right\}^\alpha \end{aligned}$$

Therefore, $\frac{\partial Y_{t+1}}{\partial Y_t} > 0$, and thus $\frac{\partial Y_{t+2}}{\partial Y_{t+1}} > 0$ and generally $\frac{\partial Y_{t+n}}{\partial Y_{t+n-1}} > 0$. Therefore, redistribution at this stage of development does not enhance economic growth.

Proof of Proposition 5 In stage I, if $k_{t+1} \in (k^h, k^l]$, we know that $B_t^{hc} > 0$, and $B_t^{lc} > 0$, while from Proposition 2, $B_t^{hw} = B_t^{lw} = 0$. From Proposition 1, high-ability capitalists invest in human capital, $E_t^{hc} = B_t^{hc} < E_t^h$, and thus $S_t^{hc} = 0$. From Proposition 1, low-ability capitalists do not invest in education, so $E_t^{lc} = 0$, and thus $S_t^{lc} = B_t^{lc}$. From Proposition 1, high-ability workers are credit constrained, so they cannot invest in education, $E_t^{hw} = 0$, and thus $S_t^{hw} = 0$. From Proposition 1, low-ability workers do not invest in education $E_t^{lw} = 0$, thus $S_t^{lw} = 0$. In this context, aggregate physical capital is given by$K_{t+1} = \sigma(1-\mu)B_t^{lc}$. On the other hand, aggregate human capital is given by $H_{t+1} = \sigma\mu H^{hc}(\frac{B_t^{hc}}{\Omega^h}) + \sigma(1-\mu) + (1-\sigma)$. Therefore, output in period $t+1$ is given by:

$$Y_{t+1} = A\left[\sigma\mu H^{hc}\left(\frac{B_t^{hc}}{\Omega^h}\right) + \sigma(1-\mu) + (1-\sigma)\right]^{1-\alpha}[\sigma(1-\mu)B_t^{lc}]^\alpha$$

In this context, we have:

$$B_t^{lc} = \beta\left(W_t + R_t B_{t-1}^{lc} - \frac{\theta(1-\beta)}{\beta}\right) = \beta\left(I_t^{lc} - \frac{\theta(1-\beta)}{\beta}\right)$$

If an amount ε_t, that is sufficiently small, is subtracted from the period t income of low-ability capitalists and transferred to the high ability workers, such that

$\bar{I}_t^{lc} = I_t^{lc} - \sigma(1-\mu)\varepsilon_t$. Then, the aggregate human capital, after redistribution, is given by:

$$H_{t+1} = \sigma\mu H^{hc}\left(\frac{B_t^{hc}}{\Omega^h}\right) + \sigma(1-\mu) + (1-\sigma)\mu H^{hw}\left(\frac{\left[\frac{\sigma(1-\mu)}{(1-\sigma)\mu}\varepsilon_t\right]}{\Omega^h}\right) + (1-\sigma)(1-\mu)$$

Then $\frac{\partial \bar{I}_t^{lc}}{\partial \varepsilon_t} < 0$, and accordingly $\frac{\partial B_t^{lc}}{\partial \varepsilon_t} = -\beta < 0$. Similarly, as ε_t is redistributed to the high-ability workers to finance their education and taking into consideration the properties of the human capital function, then $\frac{\partial H_t^{hw}}{\partial \varepsilon_t} > 0$. The final effect on output in period $t+1$ of the redistribution is given by:

$$\begin{aligned}\frac{\partial Y_{t+1}}{\partial \varepsilon_t} &= \frac{\partial Y_{t+1}}{\partial H_{t+1}}\frac{\partial H_{t+1}}{\partial H_t^{hw}}\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{\partial L_t^{hw}}{\partial \varepsilon_t} + \frac{\partial Y_{t+1}}{\partial K_{t+1}}\frac{\partial K_{t+1}}{\partial B_t^{lc}}\frac{\partial B_t^{lc}}{\partial \varepsilon_t} \\ &= A(1-\alpha)H_{t+1}^{-\alpha}K_{t+1}^{\alpha}(1-\sigma)\mu\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{[\frac{\sigma(1-\mu)}{(1-\sigma)\mu}]}{\Omega^h} \\ &\quad - AH_{t+1}^{1-\alpha}\alpha K_{t+1}^{\alpha-1}\sigma(1-\mu)\beta\end{aligned}$$

Therefore, $\frac{\partial Y_{t+1}}{\partial \varepsilon_t} > 0$ if and only if:

$$A(1-\alpha)H_{t+1}^{-\alpha}K_{t+1}^{\alpha}(1-\sigma)\mu\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{[\frac{\sigma(1-\mu)}{(1-\sigma)\mu}]}{\Omega^h} > AH_{t+1}^{1-\alpha}\alpha K_{t+1}^{\alpha-1}\sigma(1-\mu)\beta$$

This is satisfied if:

$$(1-\alpha)k_{t+1}^{\alpha}\frac{\partial H_t^{hw}}{\partial L_t^{hw}} > \Omega^h\alpha k_{t+1}^{\alpha-1}\beta$$

This can be arranged as:

$$k_{t+1} > \frac{\alpha}{1-\alpha}\frac{\beta\Omega^h}{(\frac{\partial H_t^{hw}}{\partial L_t^{hw}})}$$

As we are focusing on $k^h < k_{t+1} \leq k^l$, or $\frac{\alpha}{1-\alpha}\frac{\Omega^h}{\gamma} < k_{t+1} \leq \frac{\alpha}{1-\alpha}\frac{\Omega^l}{\gamma}$. Then, there exists a situation where redistribution from low-ability capitalists to high-ability workers induces growth, $\frac{\partial Y_{t+1}}{\partial \varepsilon_t} > 0$, if an only if $\frac{\beta\Omega^h}{(\frac{\partial H_t^{hw}}{\partial L_t^{hw}})} < \frac{\Omega^l}{\gamma}$.

Proof of Proposition 6 For any given $B > B^l$, where B^l is independent of A, since $\beta[W(B) + R(B)B - \theta\frac{1-\beta}{\beta}]$ is strictly increasing in A, there exists a sufficiently large A such that $\beta[W(B) + R(B)B - \theta\frac{1-\beta}{\beta}] > B$. Note that $\bar{B}$ decreases with A, however a sufficiently large $\theta\frac{1-\beta}{\beta}$ ensures that $\bar{k} > k^l$. Assume that $\beta[W(\bar{B}) + R(\bar{B})\bar{B} - \theta\frac{1-\beta}{\beta}] > \bar{B}$. This implies, in the absence of investment in human capital that:

$$\beta[W(B_t^c) + R(B_t^c)B_t^c - \theta\frac{1-\beta}{\beta}] > B_t^c; \quad \text{for } B_t^c \in [B^l, \bar{B}]$$

Since $\frac{\partial B_{t+1}^c}{\partial E_t^c} > 0$ for $B_t^c \in [B^l, \bar{B}]$, and $E_t^{jc} \in [0, E_t^j]$, then $B_{t+1}^c \geq \beta[W(B_t^c) + R(B_t^c)B_t^c - \theta\frac{1-\beta}{\beta}] > B_t^c$; for $B_t^c \in [B^l, \bar{B}]$. Therefore, the dynamical system has no steady state equilibrium in this interval.

Proof of Proposition 7 In stage II, we know that $B_t^{hc} > 0$, and $B_t^{lc} > 0$, while from Proposition 2, $B_t^{hw} = B_t^{lw} = 0$. From Proposition 1, high ability capitalists invest in human capital, $E_t^{hc} = E_t^h$, and thus $S_t^{hc} = (B_t^{hc} - E_t^h)$. From Proposition 1, low ability capitalists invest in education, so $E_t^{lc} = E_t^l$, and thus $S_t^{lc} = (B_t^{lc} - E_t^l)$. From Proposition 1, high-ability workers are credit constrained, so they can not invest in education, $E_t^{hw} = 0$, and thus $S_t^{hw} = 0$. From Proposition 1, low-ability workers do not invest in education $E_t^{lw} = 0$, thus $S_t^{lw} = 0$. In this context, aggregate physical capital is given by:

$$K_{t+1} = \sigma\mu(B_t^{hc} - E_t^h) + \sigma(1-\mu)(B_t^{lc} - E_t^l)$$

On the other hand, aggregate human capital is given by:

$$H_{t+1} = \sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma)$$

Therefore, output in period $t+1$ is given by:

$$Y_{t+1} = A\left[\sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma)\right]^{1-\alpha} \times [\sigma\mu(B_t^{hc} - E_t^h) + \sigma(1-\mu)(B_t^{lc} - E_t^l)]^{\alpha}$$

If an amount ε_t, that is sufficiently small, is subtracted from the period t income of capitalists and transferred to the workers, where $B_t^{hc} - E_t^h > \varepsilon_t$, and $B_t^{lc} - E_t^l > \varepsilon_t$, then we have $\bar{I}_t^{hc} = I_t^{hc} - \mu\varepsilon_t$ and $\bar{I}_t^{lc} = I_t^{lc} - (1-\mu)\varepsilon_t$, such that $\bar{I}_t^{hw} = I_t^{hw} + \mu\varepsilon_t$ and $\bar{I}_t^{lw} = I_t^{1w} + (1-\mu)\varepsilon_t$. After redistribution, the aggregate human capital is given by:

$$H_{t+1} = \sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma)\mu H^{hw}\left(\frac{\left[\frac{\sigma}{(1-\sigma)\mu}\right]\varepsilon_t}{\Omega^h}\right)$$
$$+ (1-\sigma)(1-\mu)H^{lw}\left(\frac{\left[\frac{\sigma}{(1-\sigma)(1-\mu)}\right]\varepsilon_t}{\Omega^l}\right)$$

The final effect on output in period $t+1$ of the redistribution is given by:

$$\begin{aligned}
\frac{\partial Y_{t+1}}{\partial \varepsilon_t} &= \frac{\partial Y_{t+1}}{\partial H_{t+1}}\frac{\partial H_{t+1}}{\partial H_t^{hw}}\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{\partial L_t^{hw}}{\partial \varepsilon_t} + \frac{\partial Y_{t+1}}{\partial H_{t+1}}\frac{\partial H_{t+1}}{\partial H_t^{lw}}\frac{\partial H_t^{lw}}{\partial L_t^{lw}}\frac{\partial L_t^{lw}}{\partial \varepsilon_t}\\
&\quad + \frac{\partial Y_{t+1}}{\partial K_{t+1}}\frac{\partial K_{t+1}}{\partial B_t^{hc}}\frac{\partial B_t^{hc}}{\partial \varepsilon_t} + \frac{\partial Y_{t+1}}{\partial K_{t+1}}\frac{\partial K_{t+1}}{\partial B_t^{lc}}\frac{\partial B_t^{lc}}{\partial \varepsilon_t}\\
&= A(1-\alpha)H_{t+1}^{-\alpha}K_{t+1}^{\alpha}(1-\sigma)\mu\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{\left[\frac{\sigma}{(1-\sigma)\mu}\right]}{\Omega^h}\\
&\quad + A(1-\alpha)H_{t+1}^{-\alpha}K_{t+1}^{\alpha}(1-\sigma)(1-\mu)\frac{\partial H_t^{lw}}{\partial L_t^{lw}}\frac{\left[\frac{\sigma}{(1-\sigma)(1-\mu)}\right]}{\Omega^l}\\
&\quad - AH_{t+1}^{1-\alpha}\alpha K_{t+1}^{\alpha-1}\sigma\mu\beta - AH_{t+1}^{1-\alpha}\alpha K_{t+1}^{\alpha-1}\sigma(1-\mu)\beta
\end{aligned}$$

Therefore $\frac{\partial Y_{t+1}}{\partial \varepsilon_t} > 0$ if and only if:

$$\begin{aligned}
&A(1-\alpha)H_{t+1}^{-\alpha}K_{t+1}^{\alpha}(1-\sigma)\mu\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{[\frac{\sigma}{(1-\sigma)\mu}]}{\Omega^h}\\
&\quad + A(1-\alpha)H_{t+1}^{-\alpha}K_{t+1}^{\alpha}(1-\sigma)(1-\mu)\frac{\partial H_t^{lw}}{\partial L_t^{lw}}\frac{[\frac{\sigma}{(1-\sigma)(1-\mu)}]}{\Omega^l}\\
&> AH_{t+1}^{1-\alpha}\alpha K_{t+1}^{\alpha-1}\sigma\mu\beta + AH_{t+1}^{1-\alpha}\alpha K_{t+1}^{\alpha-1}\sigma(1-\mu)\beta
\end{aligned}$$

This is satisfied if:

$$(1-\alpha)k_{t+1}^{\alpha}\left[\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{\sigma}{\Omega^h} + \frac{\partial H_t^{lw}}{\partial L_t^{lw}}\frac{\sigma}{\Omega^l}\right] > \alpha k_{t+1}^{\alpha-1}\sigma\beta$$

which can be rearranged as:

$$k_{t+1} > \frac{\alpha\beta}{(1-\alpha)\left[\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{1}{\Omega^h} + \frac{\partial H_t^{lw}}{\partial L_t^{lw}}\frac{1}{\Omega^l}\right]}$$

since we are focusing on $k^l < k_{t+1} < \bar{k}$, there exists a situation where redistribution from capitalists to workers induces growth, $\frac{\partial Y_{t+1}}{\partial \varepsilon_t} > 0$, if an only if:

$$\frac{\beta}{\left[\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{1}{\Omega^h} + \frac{\partial H_t^{lw}}{\partial L_t^{lw}}\frac{1}{\Omega^l}\right]} < \frac{\Omega^l}{\gamma}$$

Proof of Proposition 8 A steady state is a triplet (k, B^c, B^w) such that $B^c = \phi(B^c, k) = \mu B^{hc} + (1-\mu)B^{lc}$, $B^w = \phi(B^w, k) = \mu B^{hw} + (1-\mu)B^{lw}$, and $k = k(B^c, B^w)$. If there exists a non-trivial steady state in this stage, then $(k, B^c, B^w) \gg 0$. For any k, there exists at most one $B^g = \phi(B^g, k)$; for $g \in (c, w)$. Since the function ϕ is independent of g, if there exists a non-trivial steady state then $B^c = B^w > 0$, and therefore $B_t^{hw} > E_t^h$ and $B_t^{lw} > E_t^l$, and the steady state is not in this stage.

Proof of Proposition 9 In stage III, we know that $B_t^{hc} > 0$, and $B_t^{lc} > 0$, while from Proposition 2, $B_t^{hw} > 0$, and $B_t^{lw} > 0$. From Proposition 1, high-ability capitalists invest in human capital, $E_t^{hc} = E_t^h$, and thus $S_t^{hc} = (B_t^{hc} - E_t^h)$. From Proposition 1, low-ability capitalists invest in education, so $E_t^{lc} = E_t^l$, and thus $S_t^{lc} = (B_t^{lc} - E_t^l)$. From Proposition 1, high-ability workers invest in education, $E_t^{hw} = B_t^{hw}$, and thus $S_t^{hw} = 0$. From Proposition 1, low-ability workers invest in education $E_t^{lw} = B_t^{lw}$, thus $S_t^{lw} = 0$. In this context, aggregate physical capital is given by:

$$K_{t+1} = \sigma\mu(B_t^{hc} - E_t^h) + \sigma(1-\mu)(B_t^{lc} - E_t^l)$$

On the other hand, aggregate human capital is given by:

$$H_{t+1} = \sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma)\mu H^{hw}\left(\frac{B_t^{hw}}{\Omega^h}\right) + (1-\sigma)(1-\mu)H^{lw}\left(\frac{B_t^{lw}}{\Omega^l}\right)$$

Therefore, output in period $t+1$ is given by:

$$Y_{t+1} = A\left[\sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma)\mu H^{hw}\left(\frac{B_t^{hw}}{\Omega^h}\right) + (1-\sigma)(1-\mu)H^{hw}\left(\frac{B_t^{lw}}{\Omega^l}\right)\right]^{1-\alpha} [\sigma\mu(B_t^{hc} - E_t^h) + \sigma(1-\mu)(B_t^{lc} - E_t^l)]^{\alpha}$$

Assume an amount ε_t, that is sufficiently small, is subtracted from the period t income of low- and high-ability capitalists and transferred to the low- and high-ability workers, where $B_t^{hc} - E_t^h > \varepsilon_t$, $B_t^{lc} - E_t^l > \varepsilon_t$, $B_t^{hw} + [\frac{\sigma}{(1-\sigma)\mu}]\varepsilon_t < E_t^h$, and

$B_t^{lw} + [\frac{\sigma}{(1-\sigma)(1-\mu)}]\varepsilon_t < E_t^l$. After redistribution, the aggregate human capital is given by:

$$H_{t+1} = \sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right)$$
$$+ (1-\sigma)\mu H^{hw}\left(\frac{B_t^{hw} + \left[\frac{\sigma}{(1-\sigma)\mu}\right]\varepsilon_t}{\Omega^h}\right)$$
$$+ (1-\sigma)(1-\mu)H^{lw}\left(\frac{B_t^{lw} + \left[\frac{\sigma}{(1-\sigma)(1-\mu)}\right]\varepsilon_t}{\Omega^l}\right)$$

The final effect on output in period $t+1$ of the redistribution is given by:

$$\frac{\partial Y_{t+1}}{\partial \varepsilon_t} = \frac{\partial Y_{t+1}}{\partial H_{t+1}}\frac{\partial H_{t+1}}{\partial H_t^{hw}}\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{\partial L_t^{hw}}{\partial \varepsilon_t} + \frac{\partial Y_{t+1}}{\partial H_{t+1}}\frac{\partial H_{t+1}}{\partial H_t^{lw}}\frac{\partial H_t^{lw}}{\partial L_t^{lw}}\frac{\partial L_t^{lw}}{\partial \varepsilon_t}$$
$$+ \frac{\partial Y_{t+1}}{\partial K_{t+1}}\frac{\partial K_{t+1}}{\partial B_t^{hc}}\frac{\partial B_t^{hc}}{\partial \varepsilon_t} + \frac{\partial Y_{t+1}}{\partial K_{t+1}}\frac{\partial K_{t+1}}{\partial B_t^{hc}}\frac{\partial B_t^{hc}}{\partial \varepsilon_t}$$
$$= A(1-\alpha)H_{t+1}^{-\alpha}K_{t+1}^{\alpha}(1-\sigma)\mu\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{\left[\frac{\sigma}{(1-\sigma)\mu}\right]}{\Omega^h}$$
$$+ A\,(1-\alpha)\,H_{t+1}^{-\alpha}K_{t+1}^{\alpha}(1-\sigma)(1-\mu)\frac{\partial H_t^{lw}}{\partial L_t^{lw}}\frac{\left[\frac{\sigma}{(1-\sigma)(1-\mu)}\right]}{\Omega^l}$$
$$- AH_{t+1}^{1-\alpha}\alpha K_{t+1}^{\alpha-1}\sigma\mu\beta - AH_{t+1}^{1-\alpha}\alpha K_{t+1}^{\alpha-1}\sigma(1-\mu)\beta$$

Therefore, $\frac{\partial Y_{t+1}}{\partial \varepsilon_t} > 0$ if and only if:

$$(1-\alpha)k_{t+1}^{\alpha}\left[\frac{\partial H_t^{hw}}{\partial L_t^{hw}}\frac{\sigma}{\Omega^h} + \frac{\partial H_t^{lw}}{\partial L_t^{lw}}\frac{\sigma}{\Omega^l}\right] > \alpha k_{t+1}^{\alpha-1}\sigma\beta$$

which is the same condition as in Proposition 7.

Proof of Proposition 10 In stage IV, we know that $B_t^{hc} > 0$, and $B_t^{lc} > 0$, and from Proposition 2, $B_t^{hw} > 0$, and $B_t^{lw} > 0$. From Proposition 1, high-ability capitalists invest in human capital, $E_t^{hc} = E_t^h$, and thus $S_t^{hc} = (B_t^{hc} - E_t^h)$. From Proposition 1, low-ability capitalists invest in education, so $E_t^{lc} = E_t^l$, and thus $S_t^{lc} = (B_t^{lc} - E_t^l)$. From Proposition 1, high-ability workers invest in education, $E_t^{hw} = E_t^h$, and thus $S_t^{hw} = (B_t^{hw} - E_t^h)$. From Proposition 1, low-ability workers invest in education $E_t^{lw} = E_t^l$, thus $S_t^{lw} = (B_t^{lw} - E_t^l)$. In this context, aggregate physical capital is given by:

$$K_{t+1} = \sigma\mu\left(B_t^{hc} - E_t^h\right) + \sigma(1-\mu)\left(B_t^{lc} - E_t^l\right) + (1-\sigma)\mu\left(B_t^{hw} - E_t^h\right) + (1-\sigma)(1-\mu)\left(B_t^{lw} - E_t^l\right)$$

On the other hand, aggregate human capital is given by:

$$H_{t+1} = \sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma)\mu H^{hw}\left(\frac{E_t^h}{\Omega^h}\right) + (1-\sigma)(1-\mu)H^{lw}\left(\frac{E_t^l}{\Omega^l}\right)$$

Therefore, output in period $t+1$ is given by:

$$Y_{t+1} = A\left[\sigma\mu H^{hc}\left(\frac{E_t^h}{\Omega^h}\right) + \sigma(1-\mu)H^{lc}\left(\frac{E_t^l}{\Omega^l}\right) + (1-\sigma)\mu H^{hw}\left(\frac{E_t^h}{\Omega^h}\right) + (1-\sigma)(1-\mu)H^{lw}\left(\frac{E_t^l}{\Omega^l}\right)\right]^{1-\alpha} \left[\sigma\mu(B_t^{hc} - E_t^h) + \sigma(1-\mu)(B_t^{lc} - E_t^l) + (1-\sigma)\mu(B_t^{hw} - E_t^h) + (1-\sigma)(1-\mu)(B_t^{lw} - E_t^l)\right]^{\alpha}$$

Assume an amount ε_t, that is sufficiently small, is subtracted from the period t income of the low- and high-ability capitalists and transferred to the low- and high-ability workers, where $B_t^{hc} - E_t^h > \varepsilon_t$, and $B_t^{lc} - E_t^l > \varepsilon_t$. After redistribution, the aggregate human capital remains the same as before the redistribution, since all types are already investing the optimal amount of expenditure on education. Therefore, $\partial Y_{t+1}/\partial \varepsilon_t = 0$.