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Projection of a medium

Daniel R. Cavagnaro

Institute for Mathematical Behavioral Sciences, University of California, Irvine, USA

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Abstract

Learning spaces, partial cubes, and preference orderings are just a few of the many structures that can be captured by a 'medium,' a set of transformations on a possibly infinite set of states, constrained by four strong axioms. In this paper, we introduce a method for summarizing an arbitrary medium by gathering its states into equivalence classes and treating each equivalence class as a state in a new structure. When the new structure is also a medium, it can be characterized as a projection of the original medium. We show that any subset of tokens from an arbitrary medium generates a projection, and that each state in the projection determines a submedium.

1. Introduction

A 'medium' is an axiomatized algebraic structure consisting of a set of transformations, or 'tokens', on a possibly infinite set of states. Media were introduced by Falmagne in 1997 in order to generalize a class of models of preference change (Falmagne, 1997). Although the axiomatization is algebraic, media have appealing representations as connected graphs and as well-graded families of sets, making them versatile modeling tools.

The states and tokens in a medium can be represented as the vertices and edges of a graph (Eppstein & Falmagne, 2002). The axioms defining a medium imply that the graph is connected and that all circuits in the graph are even. Furthermore, the graph of any medium can be embedded isometrically into a hypercube. The isometric embedding of graphs into hypercubes has been studied extensively since the early 1970s, beginning with the work of Graham and Pollak (1971) on finding addressing schemes for communication networks. Djoković (1973) gives an elegant characterization of the graphs with such an embedding, calling them partial cubes. Subsequently, many types of partial cubes have been classified and algorithms have been developed to recognize them (Bonnington, Klavžar, & Lipovec, 2003; Brešar, Imrich, & Klažar, 2003; Klavžar & Lipovec, 2003; Winkler, 1984). Ovchinnikov (2005) proves that a graph represents a medium if and only if it is a partial cube.

Each state in a medium can be represented by its 'token content' (Definition 4), which is a set of tokens that uniquely identifies that state. The family of sets consisting of the token content of each state in a medium is '2-graded' (Definition 8). Furthermore, any medium can be equipped with an orientation, which classfies tokens as positive or negative, and each state in an oriented medium can be identified by its 'positive content'. The family of sets consisting of the positive content of each state in a medium is 'well-graded' (Definition 8), thus any oriented medium defines a well-graded family of sets (Doignon & Falmagne, 1997). Conversely, any well-graded family of sets can be cast as an oriented medium (Falmagne & Ovchinnikov, 2002). Well-graded families (and by extension media) are of interest to theorists in several different areas of mathematical social science. The family of all partial orders on a finite set was shown by Bogart (1973) to be well-graded. Various other families of order relations, such as linear orders, semi-orders, interval orders, and bi-orders, also share this property of well-gradedness (Doignon & Falmagne, 1997; Falmagne, 1997), and, as such, can be cast as media. Such media have been utilized in political choice theory for various stochastic models of the evolution of preferences (Falmagne, 1996; Falmagne, Hsu, Leite, & Regenwetter, 2005; Falmagne, Regenwetter, & Grofman, 1997; Regenwetter, Falmagne, & Grofman, 1999). Well-graded families of sets also appear in the

E-mail address: drc@uiuc.edu

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theory of knowledge acquisition (Doignon & Falmagne, 1999). A 'learning space', which was introduced under the name 'well-graded knowledge space' by Doignon and Falmagne (1985), can be characterized as a well-graded family of sets that is ∪-closed and contains the empty set (Cosyn & Uzun, 2006). Learning spaces are applied in mathematics education. In this setting, a learning space is a family of 'knowledge states,' which are subsets of the set of problems, for example in basic statistics that a student must learn to solve in order to master the subject. Typically, the number of states in a learning space is in the millions, which raises the problem of efficiently summarizing learning spaces for storage in computer memory.

In this paper we show that any medium can be summarized by a '(subset) projection.' A projection of a medium is itself a medium. The states in a projection are equivalence classes of states from the original medium. The equivalence classes are constructed based on a fixed set Γ of tokens; two states are in the same equivalence class if and only if one can be reached from the other by the application of a finite sequence of tokens from Γ. Any fixed subset of tokens induces a projection. A projection of a medium 'lawfully summarizes' that medium in the sense that a token τ^*, in the projection, sends an equivalence class of states $[S]$ to a different equivalence class $[Q]$ if and only if the corresponding token τ in the original medium sends a member of $[S]$ to a member of $[Q]$. Furthermore, subset projections generate 'submedia' in that each state in any subset projection of an arbitrary medium determines a submedium of that medium.

Projection of a medium is a special case of a general method for simplifying media. Any medium can be summarized by partitioning its set of states into equivalence classes and treating each equivalence class as a state in a new structure, which we call a 'coarsening' of the medium. A coarsening of a medium consists of a set of equivalence classes of states, and a set of binary relations on the equivalence classes, which are constructed so that the coarsening lawfully summarizes the original medium. Any equivalence relation on the set of states of a given medium induces a coarsening of that medium. By its construction, any coarsening of a medium lawfully summarize that medium, but in most cases the coarsening itself is not a medium. We prove that a coarsening of a medium is a medium if and only if it is a subset projection of that medium. In other words, we prove that the subset projections of a medium are the only media that lawfully summarizes that medium with respect to a partitioning of its set of states.

2. Review of basic concepts

The following material is drawn from the literature (Falmagne, 1997; Falmagne & Ovchinnikov, 2002). We only include material that directly pertains to this paper. Proofs of Theorems 5, 6, and 9 can be found in the aforementioned references.

Definition 1. Let $\mathscr{S}$ be a set of states. A *token* (of information) is a function $\tau : S \mapsto S\tau$ mapping $\mathscr{S}$ into itself. We use the abbreviation $S\tau_1\tau_2\cdots\tau_N = \tau_n[\cdots[\tau_2[\tau_1(S)]]]$ to denote the function composition. We say that a token τ is *effective for* S if $S\tau \neq S$. We denote by ι the identity function on $\mathscr{S}$ which by definition is not a token. Let $\mathscr{T}$ be a set of tokens on $\mathscr{S}$. The pair $(\mathscr{S}, \mathscr{T})$ is then called a *token system*. A token $\tilde{\tau}$ is the *reverse* of a token τ if for all distinct S, $V \in \mathscr{S}$ we have

$$S\tau = V \Leftrightarrow V\tilde{\tau} = S.$$

A finite composition $\boldsymbol{m} = \tau_1\cdots\tau_n$ of not necessarily distinct tokens such that $S\tau_1\cdots\tau_n = V$ is called a *message* from S to V, and we say $\boldsymbol{m}$ produces V from S. The *content* of a message $\boldsymbol{m} = \tau_1\cdots\tau_n$, denoted $\mathscr{C}(\boldsymbol{m})$, is the set $\{\tau_1,\ldots,\tau_n\}$ of its tokens. A message $\boldsymbol{m} = \tau_1\cdots\tau_n$ is *stepwise effective* for a state S if $S\tau_1\cdots\tau_k \neq S\tau_1\cdots\tau_{k-1}$, $1 \leqslant k \leqslant n$. A message is *consistent* if it does not contain both a token and its reverse. A message that is both consistent and stepwise effective for some state S is said to be *straight* for S. Two messages $\boldsymbol{m}$ and $\boldsymbol{n}$ are *jointly consistent* if $\boldsymbol{mn}$ (or equivalently $\boldsymbol{nm}$) is consistent. A message $\boldsymbol{m} = \tau_1\cdots\tau_n$ is *vacuous* if its set of indices $\{1,\ldots,n\}$ can be partitioned into pairs $\{i,j\}$ such that one of τ_i, τ_j is a reverse of the other.

Definition 2. A token system is called a *medium* if the following axioms are satisfied:

(M1) Every token τ has a unique reverse, which we denote by $\tilde{\tau}$.
(M2) For any two distinct states S, Q there is a consistent message transforming S into Q.
(M3) A message that is stepwise effective for some state is ineffective for that state if and only if it is vacuous.
(M4) Two straight messages producing the same state are jointly consistent.

Remark 3. Axioms (M1) and (M3) imply that all circuits in the adjacency graph of a medium are even.

Definition 4. Let $(\mathscr{S}, \mathscr{T})$ be a medium. For any state S, the *content* of S is the set $\hat{S}$ of all tokens each of which is contained in at least one straight message producing S.

Theorem 5. *If S and Q are two distinct states such that $S\boldsymbol{m} = Q$ for some straight message $\boldsymbol{m}$, then $\hat{Q}\backslash\hat{S} = \mathscr{C}(\boldsymbol{m})$.*

Theorem 6. *For any token τ and any state S, we have either $\tau \in \hat{S}$ or $\tilde{\tau} \in \hat{S}$, but not both. Moreover, $S = Q$ if and only if $\hat{S} = \hat{Q}$.*

Example 7. The family of all linear orders on any finite set can be cast as a medium by treating each ordering as a state and letting the tokens consist in switching two adjacent elements in an order. Take for example the family of linear orders on the set $\{A, B, C\}$. We denote by ABC the state corresponding to the ordering in which $A > B > C$.

The action of the token τ_{ij} is to transform one linear order into another by transposing elements i and j if they are adjacent and j is preferred to i, and otherwise do nothing. In terms of token content, the effect of token τ_{ij} on state S is to remove τ_{ji} from $\hat{S}$ (if $\tau_{ji} \in \hat{S}$) and replace it with τ_{ij}. This medium, which we denote by $\mathscr{L}3$, is represented by the digraph in Fig. 1. Each vertex of the digraph represents a state, and each directed edge corresponds to a token. For example, the directed edge labeled τ_{BA}, beginning at vertex ABC and ending at vertex BAC, indicates that the application of token τ_{BA} to state ABC produces the state BAC.

Definition 8. We write $Y\Delta Z = (Y\backslash Z) \cup (Z\backslash Y)$ for the symmetric difference between two sets Y and Z. Let k be any natural number. A family of sets $\mathscr{F}$ is *k-graded* if for any two distinct sets Y and Z in $\mathscr{F}$, there exists a sequence of sets $Y = Y_1, \ldots, Y_n = Z$ in $\mathscr{F}$ satisfying

(i) $|Y_i \Delta Y_{i+1}| = k$ for $i = 1, \ldots, n-1$
(ii) $|Y\Delta Z| = kn$.

A 1-graded family is said to be *well-graded*.

Theorem 9. *The content family of the set of states in a medium is* 2*-graded. More specifically, let S and Q be any two distinct states, and let* $\mathbf{m} = \tau_1 \ldots \tau_n$ *be a straight message producing Q from S, with* $S = S_0\tau_1 = S_1, S_1\tau_2 = S_2, \ldots, S_{n-1}\tau_n = S_n = Q$. *Then for* $1 \leqslant i \leqslant n$, $\hat{S}_i \backslash \hat{S}_{i-1} = \{\tau_i\}$ *and* $\hat{S}_{i-1} \backslash \hat{S}_i = \{\tilde{\tau}_i\}$. *Moreover,* $|\hat{S}\Delta\hat{Q}| = 2n$.

Example 10. Any well-graded family $\mathscr{F}$ of subsets of a set $X = \cup\mathscr{F}$ can be represented as a medium $(\mathscr{F}, \mathscr{T})$ where $\mathscr{T}$ contains, for all $x \in X \backslash \cap \mathscr{F}$, the two transformations τ_x, $\tilde{\tau}_x$ of $\mathscr{F}$ into $\mathscr{F}$ defined by

$$\tau_x : S \mapsto S\tau_x = \begin{cases} S \cup \{x\} & \text{if } x \notin S \text{ and } S \cup \{x\} \in \mathscr{F}, \\ S & \text{otherwise,} \end{cases}$$

$$\tilde{\tau}_x : S \mapsto S\tilde{\tau}_x = \begin{cases} S \backslash \{x\} & \text{if } x \in S \text{ and } S \backslash \{x\} \in \mathscr{F}, \\ S & \text{otherwise.} \end{cases}$$

It is easy to verify that $(\mathscr{F}, \mathscr{T})$ satisfies Axioms (M1)–(M4).

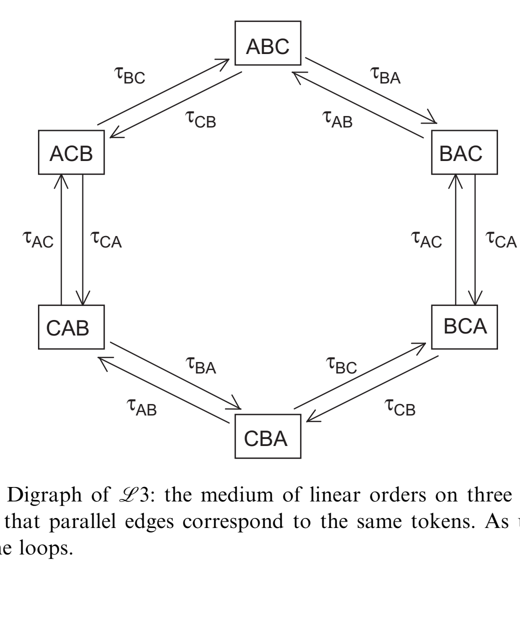

Fig. 1. Digraph of $\mathscr{L}3$: the medium of linear orders on three elements. Notice that parallel edges correspond to the same tokens. As usual, we omit the loops.

Remark 11. The set X in the preceding example need not be finite. In fact it does not even need to be countable. Consider, for example, the case where $X = \mathbb{R}$, the set of all real numbers, and $\mathscr{F}$ is the family of all co-finite subsets of $\mathbb{R}$. While this medium contains uncountably many states, each of which is a set containing uncountably many elements, any given state differs from any other state by a finite number of elements, so any state can be reached from any other state by a finite sequence of tokens.

3. Coarsenings of a token system

A coarse look at an object is a representation of that object that ignores fine details in order to emphasize broad structures. The fine details in a medium are the relationships between individual states, and the broad structures are the relationships between sets of states. The coarsening of a medium is a formal way of taking a coarse look at a medium. We coarsen a medium by partitioning its set of states into equivalence classes and constructing a set of binary relations on those equivalence classes, which capture the broad structure with respect to the partition.

Observe, for example, that we can gather states of the medium in Example 7 into the following equivalence classes:

$$[ABC] = \{ABC,\ BAC\}, \quad [CAB] = \{CAB,\ CBA\},$$
$$[BCA] = \{BCA\} \quad \text{and} \quad [ACB] = \{ACB\}.$$

We can then construct a set of binary relations that lawfully summarize the tokens in the medium with respect to our partition. That is, define

$$\tau^*_{BC} = \{([ACB], [ABC]), ([CBA], [BCA])\},$$
$$\tau^*_{CB} = \{([ABC], [ACB]), ([BCA], [CBA])\},$$
$$\tau^*_{AC} = \{([CAB], [ACB]), ([BCA], [BAC])\},$$
$$\tau^*_{CA} = \{([ACB], [CAB]), ([BAC], [BCA])\}.$$

It is easy to see that each binary relation defines a function on the set of equivalence classes. Therefore, we can view the equivalence classes as states and the binary relations as tokens in a new token system, depicted in Fig. 2(b). A quick check of the media axioms shows that this new token system is actually a medium.

Coarsening is an intuitive concept that we can define at the more general level of token systems. Any equivalence relation on the set of states of a token system defines a coarsening of that token system. We adopt the following notation.

Definition 12. Let $(\mathscr{S}, \mathscr{T})$ be a token system, and let $\approx$ be an equivalence relation on $\mathscr{S}$. We write

$$[S] = \{S' \in \mathscr{S} : S' \approx S\}$$

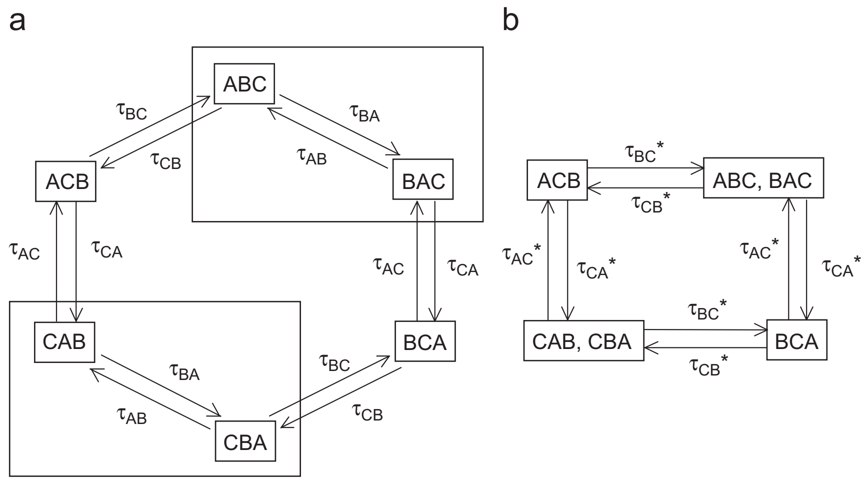

Fig. 2. (a) A partitioning of the set of states in $\mathscr{L}3$. (b) Digraph of the resulting medium. As usual, we omit the loops.

for the equivalence class containing the state S, and define

$\mathscr{S}_{\sim} = \{[S]\}_{S\in\mathscr{S}}$.

If $\tau \in \mathscr{T}$ and $S\tau \in [S]$ for all $S \in \mathscr{S}$ then we call τ an *insider* (with respect to $\sim$). If τ is not an insider then it is called an *outsider* (with respect to $\sim$). The set of all insiders and outsiders are denoted by $\mathscr{I}_{\sim}$ and $\mathscr{O}_{\sim}$, respectively. For each token $\tau \in \mathscr{O}$, the binary relation τ^* on $\mathscr{S}_{\sim}$ defined by the equivalence

$$[S]\tau^*[Q] \iff \begin{cases} [Q]\neq[S] \text{ and } \exists S' \in [S] \quad \text{such that } S'\tau \in [Q] \\ \text{or} \\ [Q] = [S] \text{ and } S'\tau \in [S] \quad \text{for all } S' \in [S] \end{cases} \tag{1}$$

is called the *summary* of τ with respect to $\sim$. We write

$\mathscr{T}_{\sim} = \{\tau^* | \tau \in \mathscr{O}_{\sim}\}$

for the set containing all of the summaries of the outsiders in $\mathscr{T}$. The pair $(\mathscr{S}_{\sim}, \mathscr{T}_{\sim})$ is called the *coarsening* of $(\mathscr{S}, \mathscr{T})$ with respect to $\sim$ (Fig. 3).

Example 13. The coarsening of a token system is not necessarily a token system because the relation τ^* defined by 1 is not necessarily a function. Consider the following partition of the set of states of $\mathscr{L}3$:

$[ABC] = \{ABC, BAC, ACB\}$,
$[BCA] = \{BCA\}$,
$[CAB] = \{CAB\}$,
$[CBA] = \{CBA\}$.

The resulting coarsening is depicted in Fig. 4. Since $BAC \in [ABC]$ and $BAC\tau_{CA} = BCA \in [BCA]$, we have $[ABC]\tau^*_{CA} = [BCA]$. But also, since $ACB \in [ABC]$ and $ACB\tau_{CA} = CAB \in [CAB]$, we have $[ABC]\tau^*_{CA}[CAB]$. Therefore τ^*_{AC} is not a function, and so the coarsening of $\mathscr{L}3$ with respect to this partition is not a token system.

Definition 14. Let $(\mathscr{S}, \mathscr{T})$ be a token system. We say that an equivalence relation $\sim$ on $\mathscr{S}$ is *token preserving* if for any token $\tau \in \mathscr{T}$ and any pair of distinct states $S, Q \in \mathscr{S}$ for which τ is effective, we have $S{\sim}Q \Rightarrow S\tau{\sim}Q\tau$.

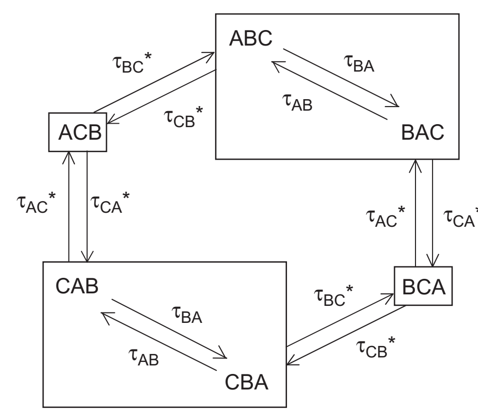

Fig. 3. Another look at the coarsening of $\mathscr{L}3$ depicted in Fig. 2(b). The summary of τ_{AB} and the summary of τ_{BA} are both the identity relation on the set of equivalence classes. Therefore, they are not tokens.

The following result is a straightforward application of Definition 14.

Theorem 15. *If* $(\mathscr{S}, \mathscr{T})$ *is a medium and* $\sim$ *is an equivalence relation on* $\mathscr{S}$ *then* $(\mathscr{S}_{\sim}, \mathscr{T}_{\sim})$ *is a token system if and only if* $\sim$ *is token preserving.*

Proof. Indeed, $(\mathscr{S}_{\sim}, \mathscr{T}_{\sim})$ is a token system if and only if τ^* is a function for all $\tau^* \in \mathscr{T}$, which is true if and only if $\sim$ is token preserving. □

4. Generating media via coarsening

The token preserving condition is not sufficient for the coarsening of a medium to be a medium.

Example 16. Let $(\mathscr{S}, \mathscr{T})$ be the medium $\mathscr{L}3$, and let $\sim$ be an equivalence relation on $\mathscr{S}$ such that $ABC{\sim}BAC$, $BCA{\sim}CBA$, and $CAB{\sim}ACB$. The coarsening of $(\mathscr{S}, \mathscr{T})$ with respect to $\sim$ is depicted in Fig. 5. It is easy to see that

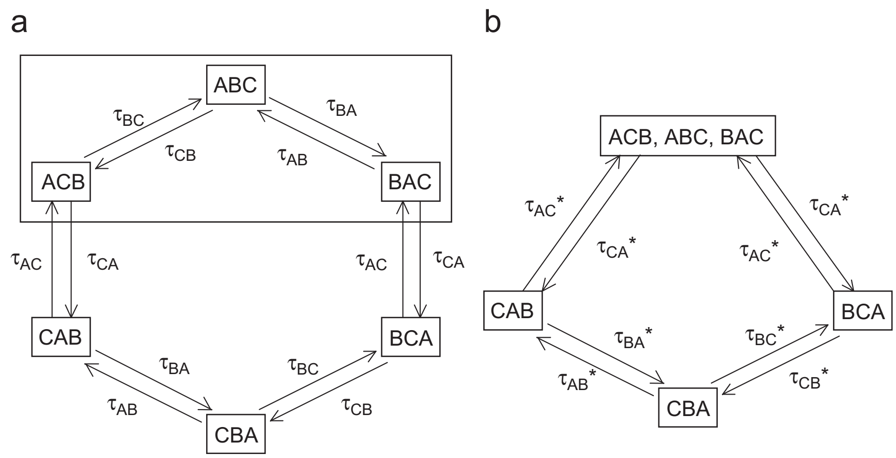

Fig. 4. (a) A partitioning of the set of states of $\mathscr{L}3$ as described in Example 13. (b) Digraph of the resultant coarsening. Notice that $[ABC]\tau^*_{CA}[BCA]$ and $[ABC]\tau^*_{CA}[CAB]$, so τ^*_{AC} is not a function and this coarsening is not a token system.

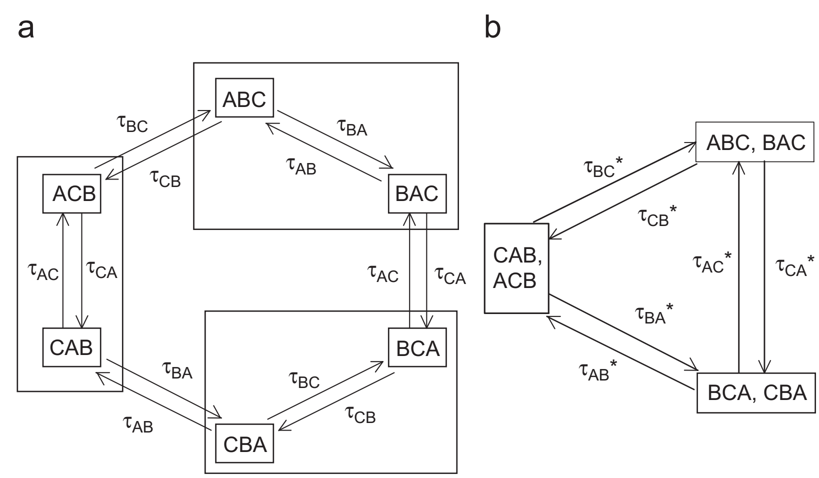

Fig. 5. (a) A partitioning of the set of states of $\mathscr{L}3$ with respect to a token preserving equivalence relation. (b) Digraph the resultant coarsening. It is easy to see that this token system is not a medium by the clear presence of odd circuits.

$\nsim$ is token preserving, but notice that $(\mathscr{S}_{\nsim}, \mathscr{T}_{\nsim})$ violates Axiom (M3) because $\tau^*_{CA}\tau^*_{AB}\tau^*_{BC}$ is a stepwise effective message for $[ABC]$ that is ineffective but not vacuous.

The following definition characterizes the partitionings of the set of states of a medium that generate media via coarsening.

Definition 17. Let $\mathscr{F}$ be a family of subsets of a universal set $\mathscr{X}$. For any subset $\Gamma \subset \mathscr{X}$ we define the binary relation $\bowtie_\Gamma$ on $\mathscr{F}$ by the equivalence

$$X \bowtie_\Gamma Y \iff X \Delta Y \subset \Gamma. \tag{2}$$

and say in this case that X is *Γ-equivalent* to Y, a term justified by the following theorem.

Theorem 18. *If $\mathscr{F}$ is a family of subsets of a universal set $\mathscr{X}$ and $\Gamma \subset \mathscr{X}$ then $\bowtie_\Gamma$ is an equivalence relation on $\mathscr{F}$.*

Proof. The relation $\bowtie_\Gamma$ is obviously reflexive and symmetric by definition. It is also transitive because, for any sets X and Y,

$$X \Delta Y \subset \Gamma \Leftrightarrow (X \Delta Y)\backslash\Gamma = \varnothing$$
$$\Leftrightarrow (X\backslash\Gamma)\Delta(Y\backslash\Gamma) = \varnothing \Leftrightarrow X\backslash\Gamma = Y\backslash\Gamma. \qquad \square$$

The equivalence relation $\bowtie_\Gamma$ can easily be applied to media by identifying states with their token contents.

Definition 19. Let $(\mathscr{S}, \mathscr{T})$ be a medium. A binary relation $\nsim$ on $\mathscr{S}$ defined by

$$S \nsim Q \iff \hat{S} \bowtie_\Gamma \hat{Q}$$

for some subset $\Gamma \subset \mathscr{T}$, where $S, Q \in \mathscr{S}$, is called a *content preserving* equivalence relation. Indeed, Theorem 18 shows that such a binary relation is an equivalence relation, and any such equivalence relation aggregates states whose token contents ‘agree’ over Γ.

Remark 20. If $(\mathscr{S},\mathscr{T})$, $\curlywedge$, and Γ are defined as above, then τ is an insider (with respect to $\curlywedge$) whenever $\tau \in \Gamma$. Thus, if $\curlywedge$ is a content preserving equivalence relation on $\mathscr{S}$, then each state $[S] \in \mathscr{S}_{\curlywedge}$ comprises all of the states in $\mathscr{S}$ that can be reached from S by a message of tokens from Γ.

Any subset of tokens from a medium defines a content preserving equivalence relation on its set of states. However, two distinct subsets induce the same partition if they have the same 'symmetric part.'

Definition 21. Let $(\mathscr{S},\mathscr{T})$ be a medium and let $\Gamma \subset \mathscr{T}$. We define the *symmetric part* of Γ to be the set $\Gamma^{\text{sym}} = \{\tau \in \Gamma : \tilde{\tau} \in \Gamma\}$, and say that Γ is *symmetric* if $\Gamma = \Gamma^{\text{sym}}$.

Theorem 22. *If $(\mathscr{S},\mathscr{T})$ is a medium and $S, Q \in \mathscr{S}$ then $\hat{S} \bowtie_{\Gamma} \hat{Q} \Longleftrightarrow \hat{S} \bowtie_{\Gamma^{\text{sym}}} \hat{Q}$.*

Proof. If $\hat{S} \bowtie_{\Gamma} \hat{Q}$ then by definition $\hat{S}\Delta\hat{Q} \subseteq \Gamma$. By Theorem 5, $\hat{S}\Delta\hat{Q} = \mathscr{C}(\boldsymbol{m}) \cup \mathscr{C}(\tilde{\boldsymbol{m}})$ where $\boldsymbol{m}$ is a straight message producing Q from S. This implies that $\hat{S}\Delta\hat{Q}$ is symmetric, so $\hat{S}\Delta\hat{Q} \subseteq \Gamma^{\text{sym}}$, so $\hat{S} \bowtie_{\Gamma^{\text{sym}}} \hat{Q}$. The converse is trivial since $\Gamma^{\text{sym}} \subseteq \Gamma$. $\square$

Remark 23. Indeed, the axioms for a medium impose symmetry in the token contents of the states in a medium, and this symmetry allows us to characterize the distinct partitions that arise from content preserving equivalence relations. By Theorem 22, there is a one-to-one correspondence between the symmetric subsets of $\mathscr{S}$ and the distinct partitions of $\mathscr{S}$ that arise from content preserving equivalence relations.

5. The Coarsening theorem

Definition 24. Let $(\mathscr{S},\mathscr{T})$ be a medium and let $\curlywedge$ be a token preserving equivalence relation on $\mathscr{S}$. Let $S, Q \in \mathscr{S}$ and let $\boldsymbol{m}$ be a message producing Q from S. We can write $\boldsymbol{m}$ as a composition of messages

$$\boldsymbol{m} = \boldsymbol{m}_0\tau_1\boldsymbol{m}_1\tau_2\boldsymbol{m}_2 \cdots \tau_n\boldsymbol{m}_n,$$

where, $\tau_i \in \mathscr{O}_{\curlywedge}$ and $\mathscr{C}(\boldsymbol{m}_i) \subseteq \mathscr{I}_{\curlywedge}$ for $1 \leqslant i \leqslant n$. The *summary* of $\boldsymbol{m}$ with respect to $\curlywedge$ is the message

$$\boldsymbol{m}^* = \begin{cases} \tau_1^* \cdots \tau_n^* & \text{if } \mathscr{C}(\boldsymbol{m}) \cap \mathscr{O}_{\curlywedge} \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases}$$

Lemma 25. *With respect to a content preserving equivalence relation on the set of states of a medium, the summary of the reverse of any token is the reverse of the summary of that token.*

Proof. Let $(\mathscr{S},\mathscr{T})$ be a medium, let $\curlywedge$ be a content preserving equivalence relation on $\mathscr{S}$, let $\alpha^* \in \mathscr{T}_{\curlywedge}$, and let $[S],[Q] \in \mathscr{S}_{\curlywedge}$ such that $[S]\alpha^* = [Q]$, i.e., there exists $S' \in [S]$ such that $S'\alpha = Q' \in [Q]$. If β^* reverses α^* then $[Q]\beta^* = [S]$, i.e., there exists $Q'' \in [Q]$ such that $Q''\beta = S'' \in [S]$. Let $\boldsymbol{m}_1$ be the straight message producing Q'' from Q' and let $\boldsymbol{m}_2$ be the straight message producing S' from S''. Then $S'\alpha\boldsymbol{m}_1 = Q''$ and $Q''\beta\boldsymbol{m}_2 = S'$. Hence $\alpha\boldsymbol{m}_1\beta\boldsymbol{m}_2$ is stepwise effective and ineffective for S', and therefore vacuous by axiom $[M3]$. Since it is vacuous and it contains α, it must also contain $\tilde{\alpha}$. By definition α is an outsider, so $\tilde{\alpha}$ must also be an outsider. It is clear that $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are composed of insiders, so neither can include $\tilde{\alpha}$.The only remaining candidate is β, so we conclude that $\tilde{\alpha} = \beta$.

Conversely, assume $\beta = \tilde{\alpha}$. If $[S]\alpha^* = [Q]$ then there exist states $S' \curlywedge S$ and $Q' \curlywedge Q$ such that $S'\alpha = Q'$. Then $Q'\beta = S'$, so $[Q]\beta^* = [S]$. Since $[S]$ and $[Q]$ were arbitrary, we conclude that β^* reverses α^*. $\square$

Corollary 26. *Let $(\mathscr{S},\mathscr{T})$ be a medium and let $\curlywedge$ be a content preserving equivalence relation on $\mathscr{S}$. If $S, Q \in \mathscr{S}$, $[S] \neq [Q]$, and $\boldsymbol{m}$ is a straight message producing Q from S, then $\boldsymbol{m}^*$ is a straight message producing $[Q]$ from $[S]$. Furthermore, if $[S],[Q] \in \mathscr{S}_{\curlywedge}$ and $\boldsymbol{m}^*$ is a straight message of tokens in $\mathscr{T}_{\curlywedge}$ producing $[Q]$ from $[S]$, then there is a straight message $\boldsymbol{n}$ of tokens in $\mathscr{T}$ such that $Q\boldsymbol{n} = S$ and $\boldsymbol{m}^* = \boldsymbol{n}^*$.*

Lemma 27. *If $(\mathscr{S},\mathscr{T})$ is a medium and $\curlywedge$ is a content preserving equivalence relation on $\mathscr{S}$ then $\curlywedge$ is a token preserving equivalence relation on $\mathscr{S}$.*

Proof. Let $\Gamma \subseteq \mathscr{T}$ such that, for any states $S, Q \in \mathscr{S}$, we have $S \curlywedge Q \Longleftrightarrow \hat{S}\Delta\hat{Q} \subseteq \Gamma$. Let $\tau \in \mathscr{T}$ and suppose τ is effective for states $V, W \in \mathscr{S}$, with $V \curlywedge W$. If $\tau \in \Gamma$ then $V\tau \curlywedge V \curlywedge W \curlywedge W\tau$. If, on the other hand, $\tau \notin \Gamma$ then

$$\begin{aligned} \widehat{V\tau} \cap \Gamma &= ((\hat{V} \cup \tau)\backslash\tilde{\tau}) \cap \Gamma = ((\hat{V} \cap \Gamma) \cup \tau)\backslash\tilde{\tau} \\ &= ((\hat{W} \cap \Gamma) \cup \tau)\backslash\tilde{\tau} = ((\hat{W} \cup \tau)\backslash\tilde{\tau}) \cap \Gamma = \widehat{W\tau} \cap \Gamma. \end{aligned}$$

In either case $V\tau \curlywedge W\tau$, so $\curlywedge$ is token preserving. $\square$

The Coarsening Theorem 28. If $(\mathscr{S},\mathscr{T})$ is a medium and $\curlywedge$ is an equivalence relation on $\mathscr{S}$ then $(\mathscr{S}_{\curlywedge},\mathscr{T}_{\curlywedge})$ is a medium if and only if $\curlywedge$ is content preserving.

Proof. (Sufficiency) If $\curlywedge$ is content preserving, then by Lemma 27, $\curlywedge$ is token preserving, so $(\mathscr{S}_{\curlywedge},\mathscr{T}_{\curlywedge})$ is a token system by Theorem 15. It remains to show that the four Axioms ([M1]–[M4]) hold for $(\mathscr{S}_{\curlywedge},\mathscr{T}_{\curlywedge})$.

[M1] This follows directly from Lemma 25.

[M2] Observe that if $[S]$ and $[Q]$ are distinct states in $\mathscr{S}_{\curlywedge}$, then S and Q are distinct states in $\mathscr{S}$ and so there exists a straight message $\boldsymbol{m}$ producing Q from S. Then $\boldsymbol{m}^*$ is a straight message producing $[Q]$ from $[S]$ by Corollary 26.

[M3] Let $\boldsymbol{m}^* = \tau_1^*, \ldots, \tau_k^*$ be a stepwise effective message producing $[Q]$ from $[S]$. By Corollary 26 there is a stepwise effective message $\boldsymbol{n}$ producing Q from S such that $\boldsymbol{n}^* = \boldsymbol{m}^*$. Suppose $\boldsymbol{m}^*$ is ineffective for $[S]$. Then $\boldsymbol{n}$ is ineffective for S, and since $(\mathscr{S},\mathscr{T})$ is a medium, that means $\boldsymbol{n}$ is vacuous. Since $\boldsymbol{n}$ is vacuous and $\boldsymbol{n}^* = \boldsymbol{m}^*$, Lemma 25 implies that $\boldsymbol{m}^*$ is vacuous. Now suppose $\boldsymbol{m}^*$ is vacuous, and let $\boldsymbol{r}$ be the straight message producing S from Q. Then $\boldsymbol{nr}$ is stepwise effective and ineffective for S, so it is vacuous. Since it is vacuous, Lemma 25 implies that $\boldsymbol{n}^*\boldsymbol{r}^*$ is also vacuous. Since $\boldsymbol{n}^* = \boldsymbol{m}^*$ and $\boldsymbol{m}^*$ is vacuous, $\boldsymbol{r}^*$ must also be vacuous. But $\boldsymbol{r}^*$ is also stepwise effective, so it is ineffective. Thus $[S]=[Q]$, meaning that $\boldsymbol{m}^*$ is ineffective for $[S]$.

[M4] Let $\boldsymbol{m}_1^*$ and $\boldsymbol{m}_2^*$ be straight messages such that $[S]\boldsymbol{m}_1^*=[Q]\boldsymbol{m}_2^*=[V]$ for some states $[S],[Q],[V]\in\mathcal{T}_{\sim}$. Then, by Corollary 26, there are straight messages $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ such that $\boldsymbol{n}_1^*=\boldsymbol{m}_1^*$, $\boldsymbol{n}_2^*=\boldsymbol{m}_2^*$, and $S\boldsymbol{n}_1=Q\boldsymbol{n}_2=V$. Axiom [M4] holds for the medium $(\mathcal{S},\mathcal{T})$, so $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are jointly consistent. Therefore $\boldsymbol{m}_1^*$ and $\boldsymbol{m}_2^*$ are jointly consistent by Corollary 26.

(Necessity) We will show that $\sim$ is content preserving by proving that $S\sim Q$ if and only if $\hat{S}\Delta\hat{Q}\subseteq\mathcal{I}_{\sim}$, for any states $S,Q\in\mathcal{S}$. By Theorem 5, it suffices to show that for any state $S\in\mathcal{S}$, $S\tau\sim S$ if and only if $\tau\in\mathcal{I}_{\sim}$. Furthermore, if $\tau\in\mathcal{I}_{\sim}$, then $S\tau\sim S$ by definition, so we only need to show that if $S\tau\sim S$ then τ is an insider. We use a proof by contradiction. Let $S\in\mathcal{S}$ and assume, for a contradiction, that $S\tau\sim S$ for some $\tau\in\mathcal{O}_{\sim}$. Since τ is an outsider, there exists $Q\in\mathcal{S}$ such that $[Q]\neq[Q\tau]$ (i.e., $[Q]\tau^*=[Q\tau]$). Let $\boldsymbol{m}$ be the straight message producing Q from S, and let $\boldsymbol{n}$ be the straight message producing $S\tau$ from $Q\tau$. By Theorem 5, neither τ nor $\tilde{\tau}$ is in $\mathcal{C}\boldsymbol{m}$ or $\mathcal{C}\boldsymbol{n}$. Taking the summary of each message yields messages $\boldsymbol{m}^*$ and $\boldsymbol{n}^*$ such that $[S]\boldsymbol{m}^*=[Q]$ and $[Q\tau]\boldsymbol{n}^*=[S\tau]$. Concatenating these messages yields a message $\boldsymbol{m}^*\tau^*\boldsymbol{n}^*$ that is ineffective for $[S]$. Removing ineffective tokens from this message yields a stepwise effective message that is ineffective for $[S]$. Since $(\mathcal{S}_{\sim},\mathcal{T}_{\sim})$ is assumed to be a medium, this message must be vacuous, but by construction, τ^* appears in this message but $\tilde{\tau}^*$ does not; a contradiction. □

6. Subset projection

The concept of coarsening a medium is useful for identifying and characterizing the media that lawfully summarize a given medium with respect to a partitioning of its set of states. However, the generality of the definition obfuscates the construction of these media and muddles the connection between the coarsenings and the media they are supposed to summarize. An alternative characterization, as projections of a given medium, makes the relevant structure of these media more salient. We offer the following definition, which is a reformulation of the definition of the coarsening of a medium in the special case in which the equivalence relation is content preserving.

Definition 29. Let $(\mathcal{S},\mathcal{T})$ be a medium and let Γ be a nonempty subset of $\mathcal{T}$. For any state $S\in\mathcal{S}$, the set

$$S_\Gamma=\{S'\in\mathcal{S}:S\boldsymbol{m}=S' \text{ for some straight message } \boldsymbol{m} \text{ with } \mathcal{C}(\boldsymbol{m})\subseteq\Gamma\}$$

is called the *augmentation* of S by Γ (or Γ-*augmentation*). Note that $\mathcal{S}$ is the augmentation of S by $\mathcal{T}$ for any $S\in\mathcal{S}$. For any token $\tau\in\bar{\Gamma}$, where $\bar{\Gamma}=\mathcal{T}\backslash\Gamma$, the function

$$\tau^*:\mathcal{S}_\Gamma\to\mathcal{S}_\Gamma:S_\Gamma\mapsto S_\Gamma\tau^*=\begin{cases}(Q\tau)_\Gamma & \text{if } Q\in[S] \text{ and } Q\tau\notin[S]\\ S_\Gamma & \text{otherwise}\end{cases}$$

is called the *summary* of τ with respect to Γ. A token system $(\mathcal{S}_\Gamma,\bar{\Gamma}^*)$ where $\mathcal{S}_\Gamma$ is the set of all distinct augmentations by Γ of states in $\mathcal{S}$ and $\bar{\Gamma}^*$ is the set of all distinct

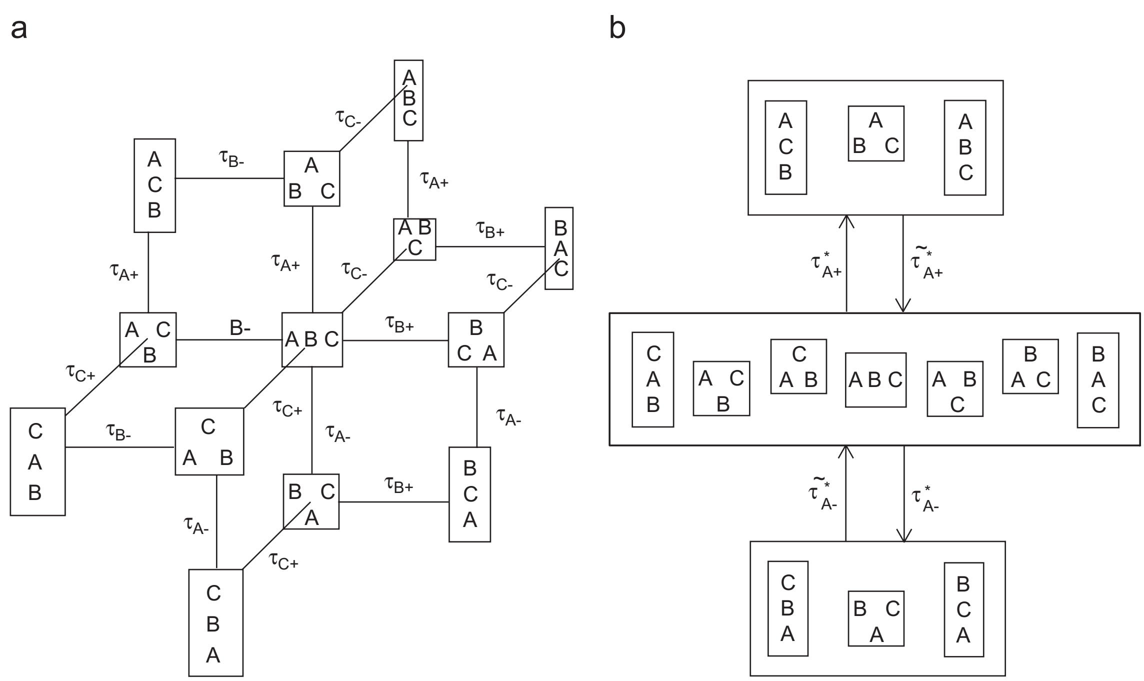

Fig. 6. (a) Adjacency graph of $\mathcal{W}3$: the medium of weak orders on three elements. (b) Digraph of the subset projection of $\mathcal{W}3$ through $\Gamma=\{\tau_{B+},\widetilde{\tau_{B+}},\tau_{B-},\widetilde{\tau_{B-}},\tau_{C+},\widetilde{\tau_{C+}},\tau_{C-},\widetilde{\tau_{C-}}\}$.

summaries of tokens in $\bar{\Gamma}$ is called the *subset projection* of $(\mathscr{S},\mathscr{T})$ (through Γ) (Fig. 6).

Combining Definition 29 with Theorem 28, we arrive at the following theorem.

The Projection Theorem 30. *Any subset projection of an arbitrary medium is a medium. Furthermore, any medium that is the coarsening of another medium is a subset projection of that medium.*

7. Subset projections and submedia

A 'submedium' is a focused representation of the local structure around a subset of the set of states of a larger medium. We borrow the following definition from Ovchinnikov (2005).

Definition 31. Let $(\mathscr{S},\mathscr{T})$ be a token system and let $\mathscr{Q}$ be a nonempty, proper subset of $\mathscr{S}$. For any $\tau \in \mathscr{T}$, the function

$$\tau_{\mathscr{Q}} : \mathscr{Q} \to \mathscr{Q} : S \mapsto S\tau_{\mathscr{Q}} = \begin{cases} S\tau & \text{if } S\tau \in \mathscr{Q}, \\ S & \text{otherwise} \end{cases}$$

is called the *reduction* of τ to $\mathscr{Q}$. A token system $(\mathscr{Q},\mathscr{T}_{\mathscr{Q}})$ where $\mathscr{T}_{\mathscr{Q}} = \{\tau_{\mathscr{Q}}\}_{\tau\in\mathscr{Q}} \backslash \{\tau_0\}$ is the set of all (distinct) reductions of tokens in $\mathscr{T}$ to $\mathscr{Q}$ different from the identity function τ_0 on $\mathscr{Q}$ is referred to as the *reduction* of $(\mathscr{S},\mathscr{T})$ to $\mathscr{Q}$. We call $(\mathscr{Q},\mathscr{T}_{\mathscr{Q}})$ a *token subsystem* of $(\mathscr{S},\mathscr{T})$. If both $(\mathscr{S},\mathscr{T})$ and $(\mathscr{Q},\mathscr{T}_{\mathscr{Q}})$ are media, then $(\mathscr{Q},\mathscr{T}_{\mathscr{Q}})$ is a *submedium* of $(\mathscr{S},\mathscr{T})$.

Subset projections and submedia are related in that each state in any subset projection of an arbitrary medium generates a submedium of that medium (Fig. 7).

Theorem 32. *If $(\mathscr{S},\mathscr{T})$ is a medium and $\Gamma \subset \mathscr{T}$, then $(S_\Gamma, \Gamma_{S_\Gamma})$ is a submedium of $(\mathscr{S},\mathscr{T})$, for any $S \in \mathscr{S}$ with $|S_\Gamma| > 1$.*

Proof. Let $(\mathscr{S},\mathscr{T})$ be a medium and let $\Gamma \subset \mathscr{T}$. The key to proving that $(S_\Gamma, \Gamma_{S_\Gamma})$ is a medium is the observation that, for any nonempty subset $\mathscr{Q}$ of $\mathscr{S}$, and any pair α and β of tokens in $\mathscr{T}$, $\beta_{\mathscr{Q}}$ reverses $\alpha_{\mathscr{Q}}$ if and only if β reverses α. With this observation in mind, we go on to show four media Axioms ([M1]–[M4]) hold for $(S_\Gamma, \Gamma_{S_\Gamma})$.

([M1]) By our observation, if $\tilde{\tau}$ is the unique reverse of τ then $\tilde{\tau}_{S_\Gamma}$ is the unique reverse of τ_{S_Γ}. By Theorem 22, we may assume that Γ is symmetric, so $\tilde{\tau} \in \Gamma$ whenever $\tau \in \Gamma$.

([M2]) Let V and W be distinct states in S_Γ. Then, by definition of S_Γ, there is a straight message $\boldsymbol{m} = \tau_1, \ldots, \tau_n$ of tokens from Γ such that $V\boldsymbol{m} = W$. Since $\tau_i \in \Gamma$ for all $\tau_i \in \mathscr{C}(\boldsymbol{m})$, we have $V\tau_1 \ldots \tau_k \in S_\Gamma$ for $1 < k \leqslant n$. Thus $\boldsymbol{m}_{S_\Gamma} = \tau_{1S_\Gamma} \ldots \tau_{nS_\Gamma}$ is a stepwise effective message of tokens from Γ_{S_Γ} such that $V\boldsymbol{m}_{S_\Gamma} = W$. By our observation, $\boldsymbol{m}_{S_\Gamma}$ is also consistent, so it is straight.

([M3]) Let $\boldsymbol{m}_{S_\Gamma}$ be a stepwise effective message for $Q \in S_\Gamma$. Then

$$\begin{aligned} &\boldsymbol{m}_{S_\Gamma} \text{ is ineffective for } Q \\ &\quad \Longleftrightarrow \boldsymbol{m} \text{ is ineffective for } Q \qquad (3) \\ &\quad \Longleftrightarrow \boldsymbol{m} \text{ is vacuous (by [M3] for the} \\ &\qquad \text{medium } (\mathscr{S},\mathscr{T})) \qquad (4) \\ &\quad \Longleftrightarrow \boldsymbol{m}_{S_\Gamma} \text{ is vacuous (by our observation).} \qquad (5) \end{aligned}$$

([M4]) Let $\boldsymbol{m}_{S_\Gamma}$ and $\boldsymbol{n}_{S_\Gamma}$ be straight messages of tokens in Γ_{S_Γ}, producing the same state in S_Γ. Then $\boldsymbol{m}$ and $\boldsymbol{n}$ are straight messages of tokens in $\mathscr{T}$, producing the same state in $\mathscr{S}$. Then $\boldsymbol{m}$ and $\boldsymbol{n}$ are jointly consistent by Axiom ([M4]) for the medium $(\mathscr{S},\mathscr{T})$, and therefore, by our observation, $\boldsymbol{m}_{S_\Gamma}$ and $\boldsymbol{n}_{S_\Gamma}$ are jointly consistent.

8. Projection of infinite media

As a final example, we offer a subset projection of a medium with uncountably many states onto a medium with countably many states. As in Remark 11, let $\mathbb{R}$ be the real numbers, and let $\mathscr{P}_{\bar{F}}(\mathbb{R})$ be the set of all co-finite subsets of $\mathbb{R}$. Each $x \in \mathbb{R}$ corresponds to two functions, τ_x and $\tilde{\tau_x}$, from $\mathscr{P}_{\bar{F}}(\mathbb{R})$ onto itself, defined by

$$\tau_x : S \mapsto S\tau_x = \begin{cases} S \cup \{x\} & \text{if } x \notin S, \\ S & \text{otherwise,} \end{cases}$$

$$\tilde{\tau_x} : S \mapsto S\tilde{\tau_x} = \begin{cases} S \backslash \{x\} & \text{if } x \in S, \\ S & \text{otherwise.} \end{cases}$$

Let $\mathscr{T} = \{\tau_x, \tilde{\tau}_x | x \in \mathbb{R}\}$ so that $(\mathscr{P}_{\bar{F}}(\mathbb{R}),\mathscr{T})$ is a medium (Remark 11). Let $\Gamma = \{\tau_y, \tilde{\tau}_y | y \in \mathbb{R}\backslash\mathbb{Q}\}$, where $\mathbb{Q}$ denotes

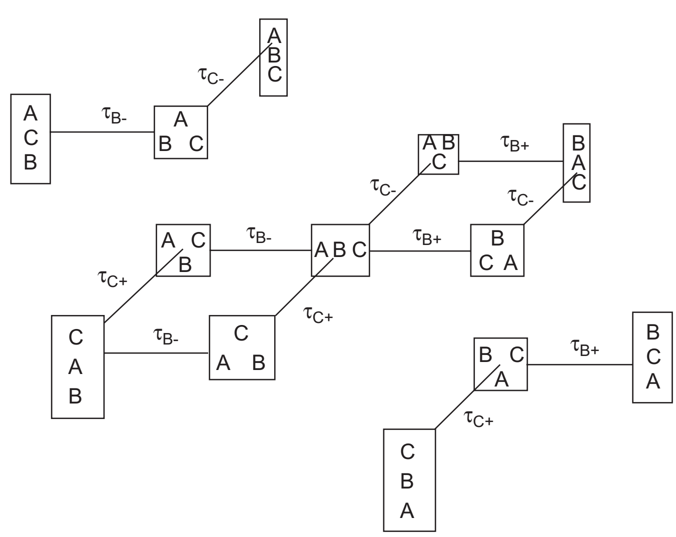

Fig. 7. Adjacency graphs of the three submedia determined by the states of the subset projection of $\mathscr{W}3$ through $\Gamma = \{\tau_{B+}, \widetilde{\tau_{B+}}, \tau_{B-}, \widetilde{\tau_{B-}}, \tau_{C+}, \widetilde{\tau_{C+}}, \tau_{C-}, \widetilde{\tau_{C-}}\}$.

the rational numbers. Then for any set $S \in \mathscr{P}_{\bar{F}}(\mathbb{R})$, the augmentation of S by Γ consists of S and each co-finite subset $S' \in \mathscr{P}_{\bar{F}}(\mathbb{R})$ that differs from S only by inclusion or exclusion of irrational numbers. Each $S \in \mathscr{P}_{\bar{F}}(\mathbb{R})$, since it is co-finite, corresponds to a finite subset of rational numbers, obtained by taking the intersection of $\bar{S}$ with the rationals. Since the states in the augmentation of S by Γ differ from one another only by irrationals, their complements also differ from one another only by irrationals, and so each $S' \in S_\Gamma$ corresponds to the same finite subset, $\bar{S} \cap \mathbb{Q}$, of rationals. Therefore, there is a one-to-one correspondents between finite subsets of $\mathscr{Q}$ and states $S_\Gamma \in \mathscr{S}_\Gamma$.

By Theorem 30, $(\mathscr{S}_\Gamma, \bar{\Gamma}^*)$ is a medium. Here, $\bar{\Gamma}^* = \{\tau_q^*, \tilde{\tau}_q^* | \, q \in \mathbb{Q}\}$. That is to say, the summary of each token in $\bar{\Gamma}$ corresponds to the inclusion or exclusion of one rational number. It is easy to check the cardinalities of the set of states and the set of tokens in this projection. Since there are countably many rational numbers, $\bar{\Gamma}^*$ is countable, and since there are countably many finite subsets of $\mathscr{Q}$, $\mathscr{S}_\Gamma$ is also countable.

By Theorem 32, $(S_\Gamma, \Gamma_{S_\Gamma})$ is a submedium of $(\mathscr{P}_{\bar{F}}(\mathbb{R}), \mathscr{T})$ for each $S \in \mathscr{P}_{\bar{F}}(\mathbb{R})$. Each token in Γ_{S_Γ} corresponds to the inclusion or exclusion of an irrational number. Therefore, by Theorem 6, each state $S \in S_\Gamma$ uniquely corresponds to a co-finite subset of irrational numbers. Since there are uncountably many irrationals, S_Γ and Γ_{S_Γ} are both uncountable.

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